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PROCEEDINGS

OF

THE PHYSICAL SOCIETY

OF LONDON.

NOVEMBER 1896.

I. *On some Experiments with Röntgen's Radiation. By* RICHARD THRELFALL, *M.A., Professor of Physics, and* JAMES ARTHUR POLLOCK, *Demonstrator of Physics in the University of Sydney, N.S.W.*[*]

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THE experiments to be described were performed during April and May of this year, and were made with the object of elucidating the nature of the radiation. It was thought that the following possible explanations should be tested:—

1. The radiation consists of a swarm of material particles projected through the glass of the generating tube. Electrical changes taking place at the glass surface are invoked to explain the differences between Röntgen's and Lenard's rays.
2. The radiation consists of an "æther wind." Æther is sucked through the glass towards the source of radiation and then blown outwards. The question as to whether the radiation observed by Röntgen is the expression of the motion of æther to or from the source remaining open.
3. The radiation consists of æther vortices moving to or from the source.

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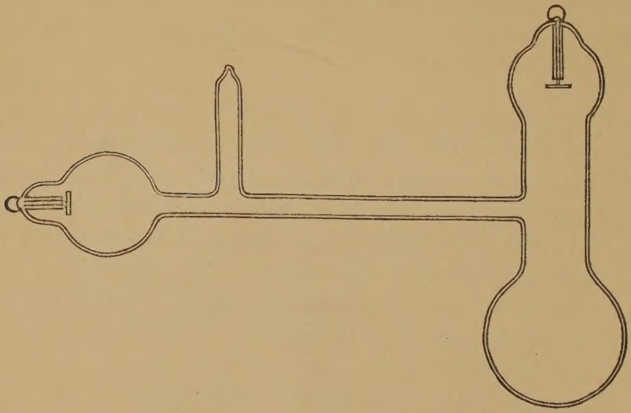
[*] Read November 13, 1896.

4. The radiation consists of æther waves: that is waves of regular or irregular æther motions.
5. The radiation consists of electromagnetic waves of either very small wave-length or having longitudinal components. This is probably a special case of (4)—at least if we are to look to the æther to explain electricity and magnetism dynamically.
6. The radiation is a phenomenon of a new order entirely unconnected in any way with anything in our past experience.

Source of Radiation.

Being thrown entirely on our own resources for means of production of the radiation—all the Crookes's tubes in our possession being almost useless—we arrived at the form of tube shown in fig. 1. These tubes are easily made; the surface opposite to the kathode being spherical, can be made very thin, and the electrodes are kept well apart. The kathode is best made very nearly plane—if concave it will easily fuse the thin glass against which its rays are projected; we have lost many tubes from this cause. We have found that the bulb may be conveniently about three or four centimetres in diameter and the main tube as little as 1·5 to 2 cm. in diameter. The expansions round the electrodes are intended to obviate

Fig. 1.



local heating, for it is not always easy to prevent oscillatory discharges and consequent "kathoding" from the "anode." The chief merits from our point of view, however, were that the tubes were very easy to make out of comparatively small glass tubing. Their volume is small, so that they can be exhausted quickly, and they give very intense action. In fact one tube—the bulb of which ultimately fused under the influence of the kathode-discharge—gave quite as intense if not more intense radiation than a "focus"-tube made in our laboratory, which appeared to act perfectly, so far as we could judge.

Experiment to test Hypothesis 1.

When a vacuum-tube is prepared with electrodes of aluminium-wire whose ends are about 1 centimetre apart, and exhausted until the discharge will rather jump across three or four centimetres of air between balls of 1·5 cm. diameter than pass in the tube, it is generally noticed that the discharge, when forced to pass by the tube, goes rather more easily in one direction than the other. By placing a spark-gap with spherical electrodes in parallel with the exhausted tube and properly adjusting the distance of the balls from one another, it is easy to so arrange matters that the sparks pass mostly by the spark-gap when the current is in one direction and by the tube when it is reversed. An arrangement of this kind is exceedingly sensitive to small changes of pressure in the exhausted tube. In the experiment to be described, the spark-gap was generally so adjusted that when the coil-commutator was in one position the whole of the discharge passed over the gap—only the faintest glow being discernible in the tube in a dark room. When the current was reversed, however, the discharge was about equally divided between the gap and the tube. No very delicate adjustment of the gap seems to be necessary, at all events when the discharges follow each other rapidly.

Having thus obtained a means of testing the vacuum in a discharge-tube more rapidly and probably much more delicately than by any kind of gauge, we thought it worth while to try whether Röntgen's rays would project particles into the exhausted tube. If hypothesis (1) be correct, then particles must be carried into the exhausted tube if it is thin enough to be transparent; and if in addition it contains a piece of platinum-foil which stops the radiation, the particles would also be stopped; also if the particles are not wholly entangled in the platinum, some change in the vacuous state of the tube is to be expected.

A tube about 12 cm. long and 1·5 cm. in diameter, and having a bulb about 4 cm. in diameter in the middle of its length, was prepared of German glass. It was provided with electrodes fused in from either end, and extending to within 1 cm. of each other in the centre of the bulb. A bit of platinum-foil lay in the bulb, and the tube was fused on to a Sprengel pump through about a metre of tubing some millimetres in internal diameter. A phosphorus-pentoxide tube was included just above the fall-tube. The whole apparatus, including the pentoxide tube, was repeatedly heated by a Bunsen burner and exhausted until the discharge preferred to pass through 3 cm. of air-gap rather than through the tube. So sensitive, however, was this means of testing the vacuum that for the first few days, despite frequent heatings and pumpings, the vacuum would not remain constant for more than some minutes. After about a fortnight of heating and pumping, however, the vacuum became so steady that the change in twelve hours, which was sufficient to entirely stop the discharge in the spark-gap, could be rectified by the fall of at most 50 drops of mercury in the fall-tube—*i. e.* by about 30 seconds' pumping. The change of vacuum occuring during an hour could still be easily detected by testing with a current in alternate directions.

Under these circumstances, experiments were made by urging Röntgen tubes to their utmost, almost in contact with the bulb of the exhausted tube, and directing their radiation on to the platinum-foil. Though many very active tubes were fused or otherwise destroyed, yet during an hour's action on several occasions no change of vacuum in the exhausted tube could be detected when due allowance for the slight progressive deterioration was made.

This experiment was repeated several times, and a tube which had successfully exhibited the fluorescence of a screen of tungstate of calcium through an aluminium plate ·7 mm.

thick to an entire audience, was melted down in the operation; but no effect whatever was observed.

Professor Wright, of Yale, has given some reasons for thinking that Röntgen rays when passing through gold-leaf carry particles (of gold?) off with them; so that if a positive effect had been obtained in the experiment described it would not have been quite conclusive; neither is the negative effect observed conclusive against *any* particles being carried forward; it only shows that the particles so carried (if any) either refuse to behave as gaseous particles, or are exceedingly few in number.

The experiments of Professor Minchin which are now available have, however, rendered the solid particle theory so unlikely that it is hardly worth while pushing the investigation further—at all events until much more powerful appliances are to hand. In case the experiment appears worthy of repetition, we would call attention to the fact that it appears desirable to use specially purified phosphorus pentoxide in the drying-tube; for the presence of the vapour of phosphorus or its lower oxide fouls the mercury when impure pentoxide is heated.

Experiments made to test whether Röntgen's Radiation is associated with æther currents in any way.

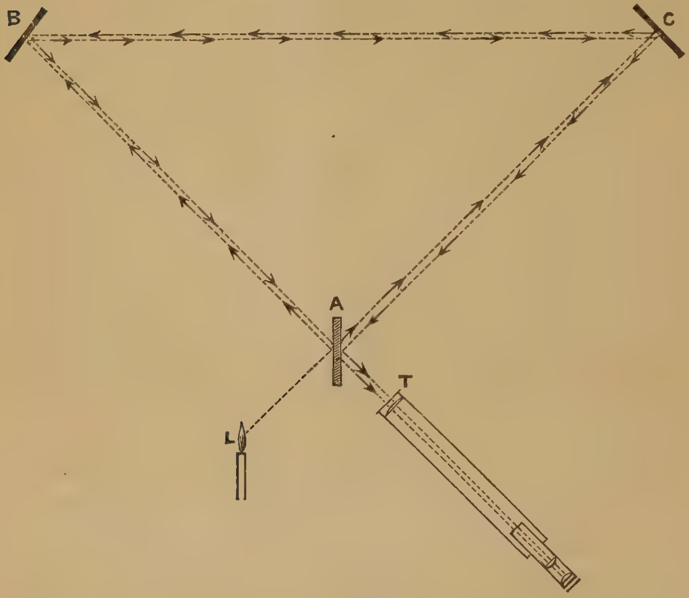
With regard to hypothesis (2) Michelson has put into our hands a comparatively simple method of obtaining the interference of two beams of light which may be used for detecting the presence of æther currents by their influence on the velocity of light travelling through the moving æther.

The arrangement employed in fig. 2 was made use of for the purpose of determining whether Röntgen's radiation is associated with æther movements.

In this arrangement the light from L is divided into two beams at A which travel round from mirror to mirror in opposite directions, eventually reaching the telescope T. When the adjustment is correct, interference-bands are seen on looking into the telescope.

In this experiment it was estimated that a shift of the bands equal to one tenth of the width of a single band, or a widening of the bands by one fifth of the same amount, could not have escaped our notice. In some experiments the Röntgen tube was placed so that the line of kathode discharge made an angle of about 30° with the path BC, and in others made an angle of 90° with the same line. No disturbance of the bands could be detected when the coil was started or while it was working, which shows at once that within the limits of accuracy imposed by the experimental conditions, the Röntgen radiation is unaccompanied by æther streams. This conclusion refers, of course, only to air, and it therefore appeared desirable to make an additional experiment, replacing the air by a substance of greater density. Pure benzene was selected as a suitable liquid, and a glass tank 16·4 cm. long, and several centimetres wide and deep, was placed in the path BC. The radiating tube, which was placed above the free surface of the benzene, was arranged to radiate on to the path in all directions, and in some cases was actually immersed in the benzene so that its active surface was in the field of view of the telescope.

Fig. 2.



No disturbance of the bands commencing when the coil started, and stopping when it stopped, was ever detected, though a great many excellent observations were made. Of course, effects due to the heating and electrification of the glass can be easily distinguished from those we are in search of in view of their persistence after the coil is put out of action. The tube employed gave quite visible fluorescence by barium platinocyanide at a metre's distance and through aluminium ·7 mm. thick. We can get an idea of the order of the minimum æther velocity which could be detected by this means. The path BC being some 25 cm. long, we will suppose that only 10 cm. of it are influenced by the tube and that the radiation is along the path. The sensitiveness of the method was the same whether we employed air or benzene.

A shift of the bands amounting to a fifth of the distance from band to band would be produced by an æther velocity sufficient to change the time of passage of light over the path by one tenth of a period. As sodium light was employed, we may call this

$$5{\cdot}9/3 \times 10^{-16} \text{ seconds.}$$

But the time required for light to travel over ten centimetres in air is about

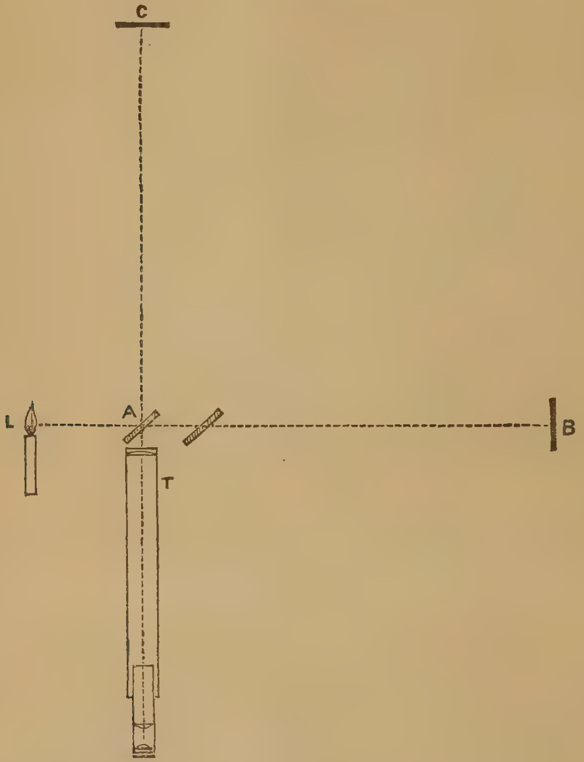
$$1/3 \times 10^{-9} \text{ seconds ;}$$

or the velocity is not changed by the radiation by more than six parts in ten million, say. This is about 177 metres per second, so that the conclusion to be drawn from the experiment is that the Röntgen radiation is not associated with æther velocities greater than, say, one fifth of a kilometre per second, or about a thousand times less than that of kathode rays as measured in a vacuous tube by Prof. J. J. Thomson (Phil. Mag. vol. xxxviii. p. 364).

Experiment to test whether Röntgen Rays produce any change in æther affecting the velocity of light.

An experiment (fig. 3) was arranged on Michelson's principles. In this case the sodium light from L is divided at the partially silvered mirror A, one beam travelling to C and thence to T, the other travelling to B and thence back to T. The path AC is operated upon by the tube, and the path AB is screened by a heavy cast-iron screen. The active surface of the tube is brought up to the path CA, and the

Fig. 3.



Röntgen radiation allowed to traverse it in a variety of directions from parallelism to normality, and is even thrown on to the mirror at C. For experiments on benzene, troughs of that liquid as similar as possible are inserted in both paths, the one in the path AB acting merely as a compensator. This experiment is much more difficult than the one previously described, and the benzene requires to be well stirred if good definition is required. In the experiments in air a shift of the bands by $\frac{1}{50}$ of the width of a band could be observed, while in benzene a change of about half the distance from band to band only could be seen. A widening of the bands to about half the above amounts could have been detected in each case. When the benzene was used the fringes were unsteady, and opportunities for observation had to be waited for. Of course, when the active surface of the Röntgen tube dipped into the benzene so as to appear in the field of view of the telescope great disturbances due to thermal and electric changes became visible. These, however, did not appear instantaneously on starting the coil, nor did they disappear on stopping it. In no case was any real effect observed.

The chief interest of this experiment lies in the fact that if the Röntgen radiation consisted of longitudinal æther waves, *i. e.*, waves of longitudinal æther displacement, some effect was to be expected. If the waves are long compared with the path AC (the path AB being screened) a widening of the fringes, or in the extreme case a total disappearance of the fringes, is to be anticipated. This can easily be realized by blowing an organ-pipe in the neighbourhood. If the waves are short compared with the dimensions AC then all will depend on the azimuth of the tube, or rather on the inclination of the Röntgen-ray path to the path of the light rays. This appears from the fact that whole waves would produce no effect in the case contemplated—everything would depend on the fractional parts of the waves included in or projected on AC.

The conclusion to be drawn from the experiment is that neither in air nor benzene are the light-transmitting properties of the æther interfered with. The limits of observational accuracy are of the same order as in the case of Experiment II.

Action of Röntgen's Rays on a Selenium Cell.

It so happened that one of us was engaged early in the year in experimenting with photo-resistance cells made of selenium which had been laboriously purified and which was probably as pure as any that has ever been obtained. Contrary to expectation, such cells showed a quite normal light-sensitiveness whether the electrodes were of platinum or of aluminium. The cells were made according to the directions given by Mr. Shelford Bidwell* in his paper read before the

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* Phil. Mag. vol. xl. p. 233; Proc. Phys. Soc. vol. xiv. p. 552.

Society last year—the only difference being that purified selenium was employed.

The result of a good many observations was to show that a certain selenium cell with platinum electrodes was acted on to about the same extent and in the same direction whether it was exposed to the radiation of a "standard" candle at a distance of three metres, or to the Röntgen radiation at a distance of ten centimetres and passing through ·7 mm. of aluminium and about 3 mm. of wood. The tube was working so as to cause visible fluorescence in a barium platinocyanide screen—not of quite the best quality—at a distance of rather over a metre in a room nearly but not absolutely dark. This is of course a very rough way of stating the degree of activity of the tube, but when the experiments were made Professor Minchin's work had not reached us, and consequently the simple scale of tube intensities which it implies was not available.

In order to test whether the action of the light differed in kind from that of the Röntgen radiation two experiments were made—in one the rate of resistance-recovery of the selenium cell was carefully studied and compared with the rate of recovery of the cell after exposure to candle-light; in the other tests were made in the hope of discovering that a permanent electromotive force was established by the radiation, and that it persisted after the radiation was cut off. Neither of these experiments led to positive results. The rate of change of resistance during the twenty seconds of exposure to Röntgen's radiation was, so far as could be seen, exactly the same as when the candle-flame was substituted at the proper distance. The recovery curves, extending over about half an hour, were also very similar on the whole (several tests were made), though both curves themselves exhibited great irregularities.

In order to obtain effects as little complicated as possible by previous history the sensitive cell was kept in circuit on the bridge, and was traversed by the testing current for two or three days before the observations.

It was for the same reason that exposures were limited to 20 seconds, for the rate of recovery of resistance with the cell employed was very slow. On one occasion when an accidental exposure for several minutes to Röntgen's radiation was made, the cell had by no means recovered four hours afterwards.

The resistance of the sensitive cell employed was reduced from 1209 ohms to 1185 ohms in 20 seconds by the radiation under the conditions mentioned. The testing battery consisted of two "Obach" cells, and the bridge was made up of two equal arms of 1000 ohms each—the selenium and the variable arm. We mention this in case it may ever turn out that the effect depends on the testing current.

With regard to the electromotive force which it was supposed might be set up. The cell was kept at rest and undisturbed for three days before the final trial; it was placed three centimetres from the active tube, which was, as before, in a metallic box, together with the coil. The tube was shut off from the cell by an aluminium plate ·7 mm. thick.

A very sensitive high resistance galvanometer in our possession, which has been described in a paper read before the Royal Society but as yet unpublished, was employed to test for any electromotive force which might be set up. An exposure to the radiation was made while the cell was in series with the galvanometer, and it was found that the cell always exhibited a small electromotive force whether it was exposed to the rays or not. This prevented the test from being very sensitive, but in no case was any electromotive force attributable to the radiation discovered, though if a voltage of 10^{-7} volts had come into operation its effect could probably not have escaped observation.

At the time these experiments were made we were uninformed as to the discharging action of the rays, which has since been so copiously studied. As soon as we saw an account of some of this work we felt that the change of resistance of the selenium cell was no longer an isolated phenomenon to be worked out by itself, but must be studied in conjunction with the similar phenomena observable in other substances, and it is for this reason that the experimental work was not extended so as to include other cells.

These notes may, perhaps, be summed up as follows:—

(1) It is easy to make a Röntgen tube of great activity by the most elementary glass-blowing.

(2) The Röntgen radiation does not consist in the projection of gaseous matter, or if it does the amount of such matter involved is extraordinarily small.

(3) The Röntgen radiation does not consist in the projection of æther streams having a velocity above a couple of hundred metres per second: this is true whether the radiation takes place in air or in benzene.

(4) The properties of æther regarded as determining the velocity of electromagnetic waves are not greatly changed (*i.e.* not at all within our experimental limits) by the Röntgen radiation, and this applies alike together in air and in benzene.

(5) A selenium cell composed of platinum electrodes and highly purified selenium is affected by Röntgen radiation to an extent which is comparable with the effect produced by diffused daylight.

(6) No permanent or temporary electromotive force is set up in a selenium cell by the Röntgen radiation.

DISCUSSION.

Mr. SHELFORD BIDWELL said he had made some experiments on the effect of Röntgen rays on the resistance of selenium, but with a negative result, although he could have detected a much smaller change than that found by the authors. It might be that this difference was due to the tube, for in his experiment the radiation started from a platinum plate within the tube, while in the authors' arrangement the radiation starts from where the kathode rays strike the glass of the tube.

Prof. SILVANUS THOMPSON said there were a number of points with reference to the Röntgen radiation which required clearing up. For instance, the suggestion that they were vortices in the æther had not been tested. Again, Lafay says that if the rays are passed through a metal screen which is charged with electricity, then the rays can be deflected by a magnet. He (Prof. Thompson) had not been able to repeat this experiment, neither had he that of Galitzine on the polarization of the rays by tourmaline. The statement of Prof. J. J. Thomson, that under the influence of the radiation paraffin became a conductor, had not been satisfactorily proved. As to the wave-length, while some observers obtained values about one-tenth that of the extreme violet, another had obtained a value greater than that of the extreme red. He (the speaker) did not understand the authors' device for detecting changes in the vacuum of a tube, since everyone who has worked with Crookes tubes has found that the resistance is always greater for a spark in one direction than in the other, and also varies with the battery-power employed. Lenard, adopting Hertz's arrangement, uses as anode a cylinder surrounding the kathode (a disk), the idea being that by using such a symmetrical arrangement the kathode radiation was more homogeneous. It might be advisable, when seeking to produce homogeneous Röntgen rays, to adopt such a symmetrical arrangement.

II. *A Satisfactory Method of Measuring Electrolytic Conductivity by means of Continuous Currents.*

By Prof. W. Stroud, *D.Sc., M.A., and* J. B. Henderson, *B.Sc.**

The devices for eliminating or reducing the disturbing effects of polarization in the measurement of the conductivity of electrolytes are very numerous. Wheatstone, Horsford, Wiedemann, Beetz, Paalzow, Ewing and Macgregor, Bouty, and others have all experimented on this subject. It will, however, be universally conceded that Kohlrausch's method of measuring the conductivity of electrolytes by the use of alternating currents and a telephone is superior to any method at present in use in which continuous currents are employed. That there are difficulties in the use of this method, arising from self-induction, capacity, &c., is admitted by Kohlrausch himself. Ever since the introduction of this method in 1875 physicists have not ceased to try to improve the continuous current methods, with the desire unquestionably of employing a galvanometer instead of a telephone as the indicating apparatus. In the method used by Fitzpatrick*, where the alternating currents are confined to the four arms of the bridge, we have an example of a successful attempt to utilize the advantages of alternating currents while dispensing with the telephone in favour of the more satisfactory galvanometer.

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* Read October 30, 1896.

In the face of all the experimental work done in the past it may seem a bold thing to say that if the proper conditions are only satisfied, continuous currents are preferable in every way to alternating currents for the measurement of electrolytic conductivity, but such we believe to be the fact.

The idea underlying the method to be described struck one of us some six months ago. This idea was to obviate the detrimental effects of polarization in the electrolytic cell by inserting a second cell with the same size of plates &c., but with a very different length of electrolytic conductor in the corresponding arm of a Wheatstone bridge circuit.

The notion of employing such a balancing cell we subsequently found was suggested in 1877 by Kohlrausch and employed by Tollinger †.

The method used by Tollinger consisted in having an electrolytic cell in each arm of the bridge, but the method of finding the resistance was to obtain a balance for a particular position of the electrodes in one cell, then to diminish the distance between them by a measured amount and increase the resistance in that arm so as to restore the balance. A conceivable source of error arises in consequence of the motion of the electrode through the liquid producing a possible alteration in its polarization.

The results of Tollinger's work show that resistances measured by his continuous current method were 0·6 per cent. lower on the average than the same resistances measured by the alternating current method.

Elsas ‡ has also employed the balancing cell method. He used a long trough of known section with two fixed non-polarizable electrodes at the ends (Cu electrodes in $CuSO_4$) and one movable electrode between them. A resistance in series with the shorter column of liquid was adjusted to balance the bridge. The resistances so measured were of the order of 12 ohms or less. For many purposes the use of non-polarizable electrodes is impossible or inconvenient.

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* Brit. Assoc. Rep. 1886, p. 328.

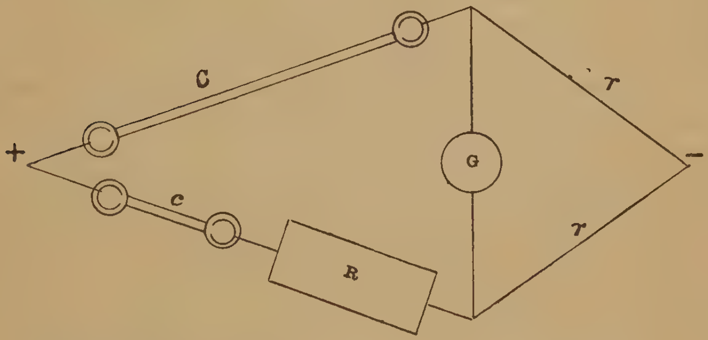
† Wied. *Ann.* vol. i. 1877, p. 510.

‡ Wied. *Ann.* xliv. p. 666.

The method to be described is in many respects similar to those of Tollinger and Elsas, but it differs from them in the form of electrolytic cell employed, and especially in a very material point, viz.:—the employment of high voltages and high resistances so as to effectually drown any residual error arising from differential polarization.

Fig. 1 will make the arrangement clear, *r r* are equal resistances, C *c* are two similar electrolytic cells equal in all respects, except that in C the electrolytic conductor is very long, in *c* very short. The resistance-box R is adjusted till there is a balance, when, of course, the resistance in the box is equal to the difference in the resistances between the two cells C and *c*. Since the resistances *r r* are equal it will be clear that when approximate balance has been obtained equal currents will be traversing both electrolytic cells, and therefore there should be the same polarization in each cell, and these polarizations are clearly opposed and are in theory eliminated.

Fig. 1.



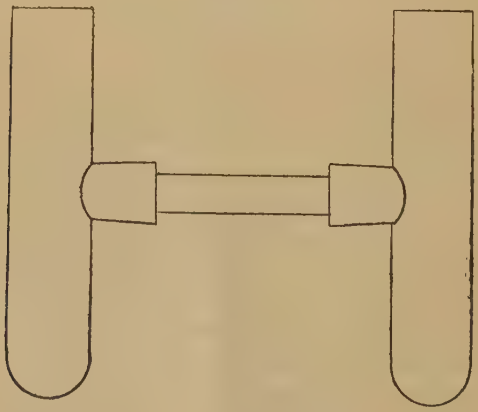
Instead of the arrangement shown in fig. 1, in which the cells are arranged in parallel, we have also tried the effect of interchanging the battery and galvanometer so as to arrange the cells in series. In the first case the polarization of one cell is opposed to that of the second, so that the polarization of each cell is much more persistent; in the other case the cells are joined in series, so that the polarization falls as soon as the battery circuit is broken. Experiment shows that the arrangement shown in the figure is much the better of the two.

In practice, however, it is found that the polarization in the two cells is not always exactly the same; generally speaking at least 99 per cent. of the polarization may be balanced in this manner, but ordinarily the balance is much more perfect than this. The next point then is to drown the residual differential polarization by working with high resistances and high voltages, and in this way the detrimental effects of polarization are reduced to insignificance. With electrolytic cells having a difference in resistance of about 20,000 ohms, the remaining arms being 1000 ohms each, and with a D'Arsonval galvanometer of about 300 ohms, when the voltage on the bridge is 30 the measurements are very like measurements of metallic resistances.

To get the specific resistance immediately in ohms, without reference to any other electrolyte, the cells C and *c* are constructed as shown in the figure.

Each cell consists of three parts, two being small, thick-walled test-tubes with necks half-way up their sides. Into these necks fit the well-ground ends of a tube of nearly uniform bore. In the case of the cell C this tube is 30 centim. or more in length, while in the case of *c* it is only a few

Fig. 2.



centim. long. In every other respect the two cells are as nearly as possible alike.

The vertical limbs of the H cells were about 1·2 centim. in diameter, and about 6 centim. high, and the horizontal tubes about 0·6 centim. external diameter, the diameter of the bore being chosen to give a convenient resistance with the electrolyte used. The lengths of the tubes were about 30 centim. in the one case and 5 centim. in the other. The tubes were calibrated by measuring their lengths and weighing in a watch-glass the mercury required to fill them. In this way a constant can be obtained for a particular pair of tubes such that on multiplying by the observed resistance of the electrolyte its specific resistance is at once obtained.

In determining the weight of mercury required to fill the tubes we found that the results were most concordant when the finely ground ends of the tube were stopped by means of a small piece of thin cover-glass backed with cork about $\frac{1}{10}$ inch thick, the diameter of the cover-glass and cork being about the same as the external diameter of the tube, so that the pressure of one finger might suffice to keep the cover-glass firmly against the end of the tube. In this way any tilting of the cover-glass relatively to the end of the tube was avoided.

The following weighings of mercury and watch-glass for one pair of tubes were made: 29·836, 29·837, and 29·835 grms. for the long tube, and 11·482, 11·481, and 11·481 for the short one. The lengths of the tubes were 29·70 and 4·89 centims. From these data, assuming the density of mercury to be 13·558 at the temperature of the experiment, the constant for this particular pair of tubes was $\frac{1}{454 \cdot 6}$.

If we were making fresh cells we should make them with the cross-tube near the foot instead of in the middle of the tubes. This would much facilitate mixing when a portion of the electrolyte is removed and replaced by water so as to alter the concentration. All that would then be necessary would be to tilt the cell two or three times to ensure effective mixing. The electrodes were pieces of platinum foil bent into cylindrical form to fit the vertical tubes of the cells. A platinum wire was welded on to each electrode and electrical connexion established with the arms of the bridge by the intervention of a mercury cup. By this means the electrodes were easily removable from the cells.

The cells and mercury cups were mounted on a wooden stand placed in an oil bath, oil being used as it was found that with a water bath the apparent resistance of the electrolyte depended on the direction of the current. The explanation of this was traced to leakage over the glass surface and through the water, and was entirely obviated by the substitution of an insulating liquid for the fluid in the bath.

The following tables give the results of some experiments made with a solution of potassium chloride, the strength being $\frac{1}{5}$ molecular equivalent per litre. The arms $r\,r$ of the bridge were each 1000 ohms, 30 volts were used, and the balancing resistance R was of the order of 20,000 ohms. Between each two measurements of resistance the electrodes were washed, heated to redness, replaced, and the current reversed. The method of making an experiment was as follows :—The current was made for about half a minute to polarize the plates, and then broken ; two minutes were allowed to elapse to enable the electrolyte to assume the temperature of the bath, which was constantly stirred, and then the balance was obtained by momentarily making the current and adjusting the resistance.

Solution I.—The solution was made from KCl twice recrystallized from water and dried in the air between filter-paper. With Kohlrausch's notation we found

$$\frac{k}{\mu}=1005{\cdot}5\times 10^{-8}.$$

We give no details of the experiment with this solution because the atomic weights used in making up the solution were only approximate, viz. :—K=39 and Cl=35·5. Correcting for this error the result would come about 0·1 per cent. higher. The salt, too, was dried in air, and it is probable that all the moisture was not removed, and any error resulting from this would cause the value obtained to be too low.

Solution II. (prepared for us with great care by Dr. Ewan).—The salt was prepared by fusing $KClO_3$ in a platinum crucible until no more gas was evolved. It was then recrystallized from dilute hydrochloric acid and dried by heating over a bunsen. The molecular weights used were K=39·13 and Cl=35·45. The strength was as before $\frac{1}{5}$ molecular equivalent per litre.

TABLE I.

Direction of current.	Temp.	Resistance.	Resistance reduced to 18° C.
+............	18·26	21035	21149
−............	18·19	21045	21129
+............	18·17	21045	21119
−............	18·11	21065	21113
+............	18·08	21085	21120
−............	18·06	21095	21121

The temperature coefficient used in the reduction was taken as 2·1 per cent. per 1° C.

The galvanometer gave a clear indication for a variation of 10 ohms.

$$\text{Specific Resistance} = \frac{21125}{454 \cdot 6} = 46 \cdot 47 \text{ ohm-centim. units.}$$

Specific Conductivity in C.G.S. units $= 2151 \cdot 9 \times 10^{-14}$.

Conductivity compared with mercury $= 2024 \cdot 4 \times 10^{-9}$.

$$\frac{k}{\mu} = 1012 \cdot 2 \times 10^{-8}.$$

Solution III. (prepared by Dr. Ewan).—The details of preparation of this solution were similar to those employed in the previous case, except that the salt was not recrystallized from dilute hydrochloric acid, so that it is possible that a trace of chlorate or perchlorate might have been present, and this would tend to diminish the conductivity.

TABLE II.

Direction of current.	Temp.	Resistance.	Resistance reduced to 18° C.
+…………	18·11	21155	21204
−…………	17·98	21225	21214
+…………	17·84	21275	21204
−…………	17·70	21345	21211
			Mean 21208

whence as before $\frac{k}{\mu}=1008{\cdot}3$.

Kohlrausch's latest result for the same strength of solution of the same salt at the same temperature is 1009, Bouty's value is 1035, Krannhal's 1003 (obtained by graphical interpolation from the data given in Ostwald's *Lehrbuch der allgemeinen Chemie*, p. 732).

Let us now investigate the different errors which are present in the determination, and assign to each its relative importance. Taking the numbers given in Table I., the mean error of observation comes to be $\pm 4{\cdot}8$, and the probable error $\pm 3{\cdot}2$. The mean error of a single observation is 12, and the probable error of a single observation is 8, so that taking only one determination, R is probably correct to 1 in 2000 or $\frac{1}{20}$ per cent. The probable error of the mean is about 1 in 7000.

Another source of error would arise if the heat caused by the passage of the current through the electrolyte had not sufficient time to escape into the liquid in the bath. The current in our experiments would produce a rise of temperature of $\frac{1}{2}°$ C. per minute, but the surface exposed is so enormous compared with the mass of electrolyte in the cross-tube that calculation shows that the difference in temperature between bath and electrolyte after vigorous stirring for two minutes is infinitesimal.

The temperature of the bath was read by means of a thermometer graduated in tenths of a degree, and standardized by comparison with a thermometer certified by the Reichsanstalt. It was, however, difficult to read the thermometer with certainty to less than $\frac{1}{50}$° C. The error arising from this cause might amount to $\frac{1}{25}$ per cent.

We estimate the possible error in the calibration of the tubes to amount to not more than $\frac{1}{20}$ per cent., but this source of error arises largely from the fact that we had not in our possession apparatus for measuring the lengths of the tubes to less than 0·01 centim. With suitable apparatus there would be no difficulty in determining the constant for the tubes with much greater accuracy.

The greatest source of error appears to be a chemical one, viz.:—the preparation of the solutions, for while the one solution prepared in one way by Dr. Ewan gave 1008·3 for the conductivity, the second prepared by a different method gave 1012·2.

We have also made experiments to see whether the alternating current method of measuring electrolytic resistance would be improved by adopting the balancing-cell which we have described. To express the results very briefly, we find that for resistances not greater than 1000 ohms or thereabouts it is distinctly an improvement, enabling dead silence to be obtained in the telephone, while without the balancing-cell there was always a feeble buzz. For resistances much greater than 1000 ohms the introduction of the balancing-cell seemed of no avail whatever. We did not pursue these experiments with alternating currents very far, because it seemed clear that there was very little chance of getting as great accuracy as with the continuous current method described.

Conclusions.

1. The form of electrolytic cell described is very suitable for determining the specific conductivity of electrolytes directly without reference to the physical properties of any second electrolyte.

2. The method of measuring the conductivity of electrolytes described in this paper is, in our opinion, more convenient and more accurate than the method in which alternating currents are used.

Discussion.

Prof. Perry asked if the authors had tested whether the difference in resistance of the two cells was proportional to the difference in length of the liquid columns.

Mr. Appleyard said he had found that the resistance of an electrolyte appeared to vary, because in the ordinary arrangement the cell was short-circuited through the arms of the bridge. He suggested as a remedy the making and breaking of the circuit by a special key, so arranged that except when taking a reading the cell is on open circuit.

Mr. Blakesley asked if the authors had tried the method in which the resistances are adjusted till, when the battery-circuit is broken, there is no immediate change in the galvanometer deflection. It is possible by this method to measure a resistance of between 6,000 and 10,000 ohms to within 0·1 per cent.

Prof. Ayrton said the method referred to by Mr. Blakesley was the ordinary "false zero" method. In using this method you were working to a continuously altering zero; in Prof. Stroud's method, however, the zero was constant.

Mr. Appleyard said he had found the "false zero" method troublesome to use.

Prof. Stroud, in reply, said they had not tested the proportionality between the resistance and length, and they had not tried the "false zero" method.

III. *Absorption of Electric Waves along Wires by a Terminal Bridge.* *By* Edwin H. Barton, *D.Sc., F.R.S.E., Senior Lecturer in Physics at University College, Nottingham, and* Geo. B. Bryan, *B.Sc., "1851 Exhibition" Research Scholar*[*].

In the discussion on Mr. Yule's electrical paper last year[†], one of us stated his intention of endeavouring to realise experimentally the extinction of electric waves along wires on their arrival at the end of the line, the urgent desirability of this having been felt by several investigators[‡]. The question is probably of interest also in connexion with Telephony. This task, small as it may seem, we had no opportunity to attack till this summer. The adoption of a resistance-bridge for the absorption of the electric waves was originally suggested by Mr. Oliver Heaviside's mathematical proof[§] that, given a bridge of suitable resistance at the end of a line, then the waves arriving there would be immediately absorbed; "the electricity is all gobbled up at once, so to speak"[‖]. And, a few preliminary difficulties being overcome, this important theoretical conclusion readily received experimental confirmation.

Experimental Arrangement.—The electric vibrations were generated by an oscillator of the type and size previously used by Mr. V. Bjerknes[¶] and others. The waves, about 8½ metres long, were propagated along a "line" consisting of a pair of parallel copper wires 1·5 mm. in diameter and 8 cm.

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[*] Read November 13, 1896.

[†] G. U. Yule, "On the Passage of an Oscillator Wave-Train through a Plate of Conducting Dielectric," Proc. Phys. Soc. vol. xiii. pp. 358–392 (1895); Phil. Mag. [5] xxxix. p. 309.

[‡] See *e.g.* J. von Geitler, pp. 7–8, 'Inaugural Dissertation,' Bonn, 1893; E. H. Barton, pp. 70–71, Proc. Roy. Soc. vol. lvii. 1894; G. U. Yule, Proc. Phys. Soc. *t. c.* pp. 384–387; Phil. Mag. *t. c.* pp. 334–337.

[§] 'Electrical Papers,' vol. ii. pp. 127 and 132–133.

[‖] *Loc. cit.* p. 127.

[¶] V. Bjerknes, Wied. *Ann.* vol. xliv. pp. 519–520 (1891); E. H. Barton, Proc. Roy. Soc. vol. liv. p. 86 (1893); G. U. Yule, Proc. Roy. Soc. vol. liv. p. 97 (1893).

apart. The waves were detected and relatively measured by an attraction-electrometer made on the principle of that devised by Herr von Geitler[*]. The length of the line to the electrometer was 116 m. This instrument has a needle of aluminium foil suspended by a quartz fibre between disks which may be connected to the respective wires of the line. On the passage of a wave-train the needle, initially uncharged, is charged inductively, and its ends are accordingly attracted to the disks whatever the sign of their potential-difference.

The electrometer therefore responds by a deflexion of the needle towards its disks. Let us call this a positive deflexion. In addition to the needle just described, and fixed to it in the same vertical plane, is a second similar one, but with its pair of attracting disks so situated that when they alone are connected to the line a negative deflexion ensues. When both needles have their respective disks connected to the line the electrometer is said to be used differentially. When so used, and with the instrument in adjustment, no deflexion is produced by the simple passage of a wave-train. Suppose now, owing to anything on the line which causes reflexion, stationary waves are produced at the electrometer. Then, provided one needle be connected to a point on the line which is a *node* for the waves, and the other be at a *loop*, the electrometer must give a deflexion and thus prove that *reflexion* of the waves has occurred. It is easily seen that when a single needle is used the first throw is proportional to the time-integral of the square of the potential-difference of the two leads of the line at the place to which the disks are connected.

The electrometer-needle carried a light plane mirror, and the throws were read by a distant telescope and scale.

Theory of Absorption.—Mr. Heaviside's[†] theory shows that in the case of plane electric waves proceeding along parallel leads to a bridge at the end of the line, the ratio of the potential-differences of the two leads due respectively to the reflected and incident waves is given by

$$\rho = \frac{\mathrm{R} - \mathrm{L}v}{\mathrm{R} + \mathrm{L}v}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

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[*] Wied. *Ann.* vol. xlix. p. 188 (1893).

[†] 'Electrical Papers,' vol. ii. pp. 132–133.

where ρ is the ratio in question, or *reflexion coefficient*, R is the resistance of the bridge, which must be of negligible inductance, L is the inductance per unit length of the line, and v is the velocity of light. Electromagnetic units are to be understood throughout.

Thus,

$$\rho=-1, \quad \text{for } R=0, \quad \ldots \quad (2)$$
$$\rho=+1, \quad \text{for } R=\infty, \quad \ldots \quad (3)$$
$$\text{and } \rho=\ 0, \quad \text{for } R=Lv. \quad \ldots \quad (4)$$

Now $v=3\times10^{10}$ cm. per sec. nearly, and in the present case $L=4\log_\epsilon\left(\frac{8}{0{\cdot}075}\right)=18{\cdot}68$ nearly. Hence the critical value of R to give complete absorption is,

$$R=3\times10^{10}\times18{\cdot}68 \text{ absolute units of resistance}=560 \text{ ohms.} \quad (5)$$

Details of Bridge. At first an ordinary resistance-box was used as a bridge across the two wires of the line. But this was soon rejected lest the faces of the two bars at a plug opening should act as the plates of a condenser of appreciable capacity, which is inadmissible in the bridge desired. We next wound a small coil non-inductively on a bobbin in the usual manner and used it without any box or screw terminals. It was of well-covered wire and further insulated by thorough soaking in paraffin-wax both before and after winding. But this too proved unsatisfactory. For a time it stood, but then the insulation broke down and sparks were seen to pass between different turns of the wire. It was accordingly abandoned.

The simple expedient of making an electrical resistance by rubbing a common lead pencil on a disk of ground glass, small pieces of tin-foil gripped on by binding-screws forming the terminals, was then tried, and seemed in every way satisfactory. The electrostatic capacity of the terminals was proved to be negligible by testing such a disk and screws without any pencil. In this case no reflexion of the waves could be detected. Moreover, by making the track of the pencil-marks between the terminals very wide, the self-induction of the bridge was reduced to a minimum. It was also found easy to adjust the resistance to a few ohms and to restore it nearly to its original value after the change which spontaneously occurred in it from day to day. Three bridges of this type were used, of resistances 261, 549 to 560, and 1336 to 1355 ohms. Let these be denoted respectively by the letters F, G, and H. Finally, since the pencil-markings on the glass form an extremely thin sheet, the resistance may be assumed to be practically the same to high-frequency waves as to the steady current by which they were measured on the Post-office box.

Experiments.—The oscillator emits rapidly-damped electric vibrations, only the first dozen, say, being appreciable. We thus have, advancing along the line with the speed of light, a damped wave-train, its large end, or head, leading and its tail, after about twelve waves, being negligibly small. Suppose now one needle of the electrometer to be connected to the line and a bridge of no resistance to be placed at the end of the line a little beyond the electrometer. We shall then have, at the electrometer, stationary waves due to interference between the incident waves and those reflected at the bridge. Hence, if a series of readings be taken with the electrometer at different distances from the end of the line the throws will be found to periodically wax and wane. But when the bridge is distant a few wave-lengths from the electrometer, we have the head of the wave-train interfering with the tail only. And with somewhat greater distances between electrometer and bridge the interferences of the waves and, consequently, the waxing and waning of the electrometer-throws cease to be appreciable. Thus, if the distances between electrometer and terminal bridge are plotted as abscissæ and electrometer throws as ordinates, we should expect the experiment to yield a damped wavy curve. And this is the case, as first shown by Mr. V. Bjerknes*, and utilized by him and his successors to determine the wave-length of the oscillations in use. The result of this experiment in our case is shown by the full-line curve E on the diagram.

In the case just considered the reflexion coefficient, ρ, of the bridge is -1, as shown by equation (2). The waves are therefore unchanged in magnitude by the act of reflexion. Turn now to the general case of a precisely similar experiment

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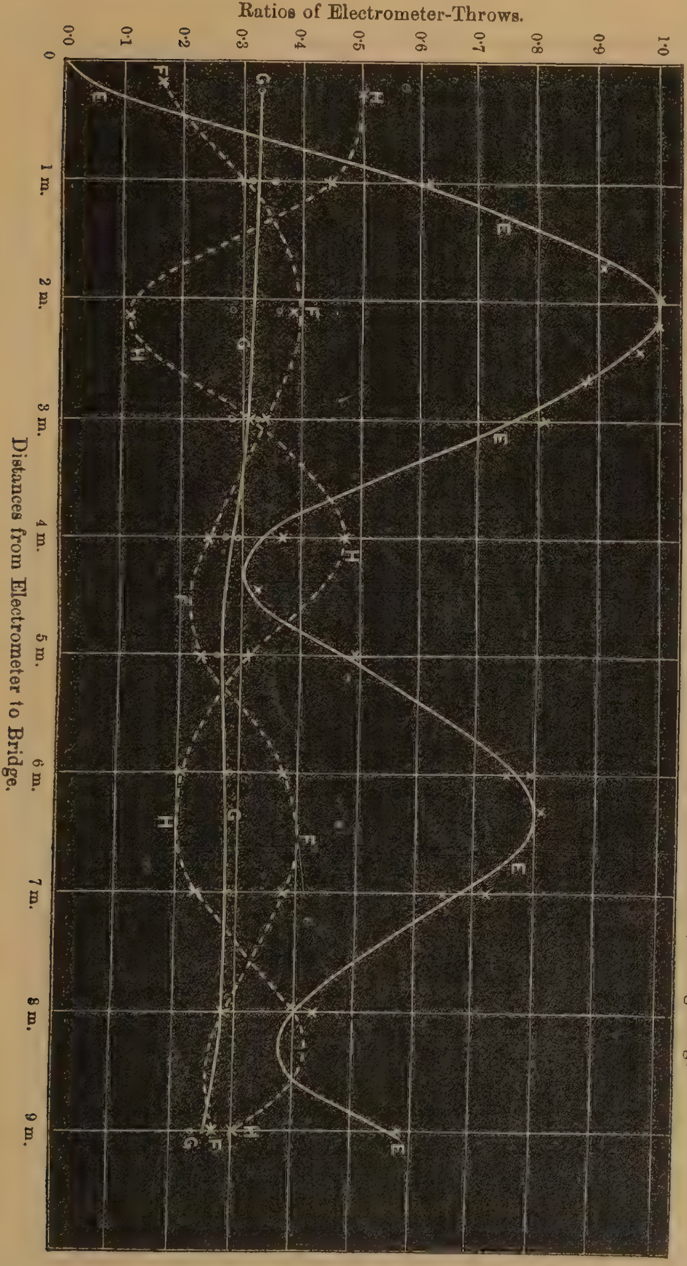
* Wied. *Ann.* vol. xliv. pp. 522–523 (1891).



Curves showing Interference due to Reflexion.

Line G G, being almost straight, indicates complete absorption by the corresponding bridge.

with a terminal bridge for which ρ is finite and less than unity. In this case, obviously, theory predicts a curve of similar form, wavy and damped, but lying between narrower limits, since the reflected wave is now always smaller than the incident one. Such an experimental result may, therefore, be accepted as proof that the bridge in use diminishes the wave in the act of reflexion.

Further,

$$\text{for } R < Lv, \ \rho \text{ has a negative value}, \quad . \quad . \quad . \quad (6)$$

and consequently the maxima and minima of the curves occur for the same values of the abscissæ as when $R=0$. This is seen to be the case with the dotted curve, F, on the diagram, obtained with the bridge whose resistance was 261 ohms.

Again,

$$\text{for } R > Lv, \ \rho \text{ has a positive value}; \quad . \quad . \quad . \quad (7)$$

hence this curve is, broadly speaking, an inversion of that just considered, the maxima occurring for those values of the abscissæ at which the minima of the curve for $R=0$ occurred, and *vice versâ*. This is exemplified in the dotted curve, H, on the diagram, resulting from the experiment with the bridge whose resistance was of the order 1350 ohms.

Finally, we have

$$\text{for } R=Lv, \quad \rho=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

In this case, since no waves are reflected, no stationary waves can obtain, and consequently no waxing and waning of the electrometer-throws can occur. Thus theory predicts a straight line as the outcome of an experiment with a bridge of this critical resistance. This is approximately the case with the full-line curve, G, on the diagram, obtained with the bridge whose resistance was of the order 560 ohms, the theoretical value of the critical resistance for the line used.

Of course an absolutely straight line cannot be expected, since the sparks in the oscillator vary slightly. For this reason, throughout the experiments hitherto considered the throws with the bridge under examination at any desired distance beyond the electrometer were alternated with those obtained with a no-resistance bridge at the standard distance of a quarter-wave-length beyond. And the ordinates of the curves on the diagram represent, not the actual electrometer-throws with the bridge and distance in question, but the ratio of these to those obtained under the standard conditions just described. Thus the maximum ordinate in the diagram is unity, all the curves are reduced to the same scale, and the errors due to variations of the sparking at the oscillator are in great measure eliminated. In the above experiments the electrometer was used as a single-needle instrument, and the throws obtained varied up to 100 scale-divisions.

As a further test of the non-reflexion of waves by the bridge G, the electrometer was afterwards used differentially, the upper and lower needles having their disks attached to points on the line a quarter-wave-length apart. The bridges were placed at the end of the line 0·25 m. beyond the second needle. With a no-resistance bridge the throws were positive and of the order 50 scale-divisions; with a simple disconnexion ("infinity bridge") the throws were of the same order but negative. The bridges F and H gave smaller throws, positive and negative respectively. The bridge G also appeared to yield a small throw; but this was no larger than might be due to a creep of the zero or an imperfection in the adjustment of the electrometer-needles. The use of the differential electrometer is thus seen to confirm the results of the experiments conducted with a single needle.

It should be noted that this paper deals only with the absorption of waves by *terminal* bridges; no *intermediate* resistance-bridge will absorb all the waves arriving there. The reflexion and transmission coefficients ρ' and τ' for an intermediate bridge of resistance R are given by*

$$\rho' = \frac{-\mathrm{L}v}{2\mathrm{R}+\mathrm{L}v} \text{ and } \tau' = \frac{2\mathrm{R}}{2\mathrm{R}+\mathrm{L}v}, \quad . \quad . \quad . \quad (9)$$

the notation being the same as that previously used. Thus the condition for non-reflexion is $\mathrm{R}=\infty$, in which case all is transmitted. Hence complete absorption is impossible.

If, therefore, we wish to pass electric waves along wires through any layer of dielectric, or through any other arrangement which affects them, and are desirous of avoiding the annoying disturbance due to return of the transmitted portion after reflexion at the end of the line, the following method will suffice.

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* Oliver Heaviside's 'Electrical Papers,' vol. ii. pp. 141–142.

1. Place at the end of the line a bridge of the form herein described, whose resistance has the value given by the theory for plane waves.

2. Test experimentally the absorbing-power of the bridge, and, if necessary, adjust its resistance until no reflexion can be detected.

University College, Nottingham,
September 1896.

Discussion.

Mr. Campbell asked if the resistances given were expressed in ohms or in electromagnetic units.

Mr. Bidwell asked if the authors had found that the pencil-trace resistances obeyed Ohm's law. He found that if you balance with one cell in the battery circuit, then, on increasing the battery-power to two cells, the resistance alters.

Mr. Appleyard suggested that the variation was caused by the contacts at the ends not being good.

Mr. Campbell said the same variation occurred in the case of mixtures of clay and plumbago where the contacts could be made quite good.

Mr. Carter suggested electroplating the ends to give good contact.

Mr. Bryan, in reply, said that in their case they did not require to know the resistance very accurately.

IV. *Applications of Physics and Mathematics to Seismology.*

By C. CHREE, *Sc.D.*[*]

3 of

Sections CONTENTS.

1–3. Introductory.
4. Influence of surface-load on observed level.
5. Direct pressure effect.
6–7. Relation between pressure and gravitational effects.
8–10. Pure pressure effects, general formulæ, application to rectangle.
11–13. Special cases of loaded rectangle.
14–18. Numerical results, and conclusions.
19–20. Pressure effects below the surface.
21–23. Luni-solar effects, explanatory.
24–26. Solution of problem.
27–30. Application to earth, working formulæ.
31. Numerical estimates.
32. Final conclusions.
33–34. Subsidiary remarks.

Introductory.

§ 1. THE existence of apparent movements in the earth's surface-strata appeals in the first instance to seismologists. Prof. J. Milne, however, and others have been of late attempting to bring it home to astronomers and meteorologists that they too may have a vital interest in the matter. The presentations of the subject which have come under my notice take little or no heed of the theoretical aspects of the case, in which, as an elastician, I have long been interested. As the neglect of theoretical results may be due not so much to their defects as to the slowness with which mathematical investigations become generally known, I have decided to group together and discuss in a more or less popular way the theoretical conclusions which seem to me the most closely connected with the subject in question.

§ 2. The mathematical work by which these conclusions were deduced refers to material which is homogeneous, isotropic, and elastic, while the body in whose phenomena the seismologist is interested is the earth.

Now it must not be supposed that I fail to appreciate the differences between the material of theory and that of nature.

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[*] Read December 11, 1896.

The certainty of the departure of many of the surface-strata from the attributes ascribed to isotropic elastic solids, and the uncertainty as to the density, solidity, and other physical properties of $\frac{999}{1000}$ of the earth's mass I perfectly realize. The conditions under which the deep-seated materials of the earth exist are fundamentally different from those we are familiar with at the surface. The enormous pressure, and the presumably high temperature, very likely combine to produce a state to which the terms solid, viscous, liquid, as we understand them, are alike inapplicable. But be the state what it may, the material must respond to the action of periodic forces; such forces must produce varying strains and stresses; and these strains and stresses can hardly fail to produce effects at the surface. No numerical estimate of these effects can claim to be in any sense final, as the mathematical work by which it is evolved must depend on physical data which are at best unproved. It appears desirable, however, that such numerical estimates should be made, on the least objectionable physical basis available, if only for the reason that their existence supplies a guide and incentive to direct observation.

In my opinion, for reasons previously discussed*, the treatment of the earth as an incompressible elastic solid is exposed to perhaps a minimum of objections. Most probably the material increases in density and temperature as we approach the centre, and a treatment which assumes the material to vary with the radial distance would possess higher *à priori* claims to regard than one which treats the earth as homogeneous throughout. The problem of a gravitating mass of varying elastic properties has, however, still to be published, and as the assumed law of variation would probably be mainly guess-work, the advantages for practical ends might be less than would appear at first sight.

Variation of the material with the radial distance, I may add, could hardly affect the general character of the phenomena Surface heterogeneity, in which the material varies rapidly with latitude or longitude, is not unlikely to modify largely the magnitude of some of the results at individual stations, but is most unlikely to produce a large effect on the order of magnitude of the mean lunar or solar tidal effects at a moderate number of stations scattered over the earth's surface.

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* Phil. Mag. Sept. 1891, p. 232.

The treatment of local surface pressures in the first part of the paper is in some respects on a less uncertain basis. We can assure ourselves, if need be, of the solidity of the ground surrounding a station; and though the mathematical work treats the solid as going down to infinity, this only means in practice that the depth must be large compared to the shortest distance of the loaded area. There are reasons, however, even here for regarding the numerical results as in general but rough approximations.

They might possess high accuracy if the surface material were bare rock in horizontal strata, and the recording apparatus were supported directly on the rock; but uncertainties are introduced when the load is applied at the surface of ordinary soil, and the support of the apparatus is stonework inside a building whose foundations go to an appreciable depth.

§ 3. The observed facts on which our investigations are most likely to bear are certain slow changes in the indications of spirit-levels or delicately suspended pendulums. Some of these Prof. Milne is disposed to attribute to meteorological agencies such as rainfall or evaporation.

A relative excess of moisture to the west, say, of an observatory is, he considers, equivalent to a surface load on that side, tending to make the ground on which the building stands slope downwards from east to west. Such want of symmetry may arise from the peculiarities of the soil, or through the ground being sheltered by trees or modified by cultivation. In sunshine the shadow of the building itself, by retarding evaporation, may set up such a difference as Prof. Milne has in view.

I am not sure that an excess of evaporation from the east, say, of a building is necessarily equivalent to the withdrawal of a surface load from that side, at least to the exact extent of the surplus evaporation. The withdrawal of moisture from the soil has a decided influence on its conductivity for heat—not to speak of electricity—and so may exert a very appreciable influence on the temperature near the surface, the consequences of which it would be difficult to follow. There is also, presumably, underground circulation both of air and moisture, which may not unlikely counteract to some extent differences of surface evaporation.

Though these and other uncertainties exist, it is certainly worth while considering the numerical magnitude of the results to be expected from the agencies postulated by Prof. Milne. The theoretical results will also, I hope, suggest the way in which the best use may be made of experimental determinations of the effects of surface loading over limited areas of convenient shape.

Influence of Surface Load on the observed Level.

§ 4. This influence is not so simple as might appear at first sight. The weight of the loading material is equivalent to a pressure normal to the surface, which we suppose horizontal. But, in addition, we must allow for the fact that the gravitational attraction of the loading material slightly alters the direction of "gravity" at the surface. Consider, for example, the influence of the ordinary ocean tide at a point inland near the shore. At high tide there is on the sea-bottom a pressure exceeding the mean by an amount corresponding to the height of the water above its mean level; this will tend to make a naturally horizontal plane dip towards the sea. At the same time the surplus volume of water will give a horizontal component to what we may regard as normal "gravity" in the neighbourhood. This second effect has been called attention to, in this very case, in Thomson and Tait's 'Natural Philosophy,' art. 818, where will be found a numerical estimate for a specified set of conditions. What a spirit-level shows is the plane perpendicular to gravity—including "centrifugal force" and all disturbing forces. We are thus obliged to consider both effects before attempting numerical estimates.

Direct Pressure Effect, Fundamental Formulæ.

§ 5. In the following calculations the earth is treated as an isotropic elastic solid, principal weight being attached to the results obtained by supposing the material incompressible.

Also, as we are primarily interested in the consequences of pressure applied over limited areas, the loaded surface is treated as a horizontal plane, on the lower side of which the material extends to infinity. On these hypotheses we are enabled to make use of the very interesting and important results established by Professors Cerruti and Boussinesq.

A convenient English abstract* of Boussinesq's work is contained in Todhunter and Pearson's 'History of Elasticity,' vol. ii. part 2, arts. 1492 *et seq.*, from which the following formulæ are quoted, the only variation being the use of Thomson and Tait's notation for the elastic constants.

The origin of coordinates lies in the undisturbed surface, taken as the plane of xy, the positive direction of the z axis being downwards into the earth. The normal pressure applied to the element $d\omega$ of surface is $p\,d\omega$, where p is supposed of course a known function of x, y.

u, v, w denote the components of elastic displacement, n the rigidity, and η Poisson's ratio for the material.

The displacements at any point x, y, z in the material are as follows:

$$u = -\frac{1}{4\pi n}\left\{\frac{d^2}{dx\,dz}\iint rp\,d\omega + (1-2\eta)\frac{d}{dx}\iint \log(z+r)p\,d\omega\right\}, \qquad (1)$$

$$v = -\frac{1}{4\pi n}\left\{\frac{d^2}{dy\,dz}\iint rp\,d\omega + (1-2\eta)\frac{d}{dy}\iint \log(z+r)p\,d\omega\right\}, \qquad (2)$$

$$w = \frac{1}{4\pi n}\left\{2(1-\eta)\iint \frac{p}{r}\,d\omega + z^2\iint \frac{p}{r^3}\,d\omega\right\}. \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

Here r is the distance between the element $d\omega$ where p is applied, and the point x, y, z where u, v, w are measured. The integration extends to all parts of the surface where p differs from zero.

The simplification in the formulæ when $\eta = \cdot 5$, or the material is incompressible, should be noticed.

The *slope*—*i.e.*, inclination to the plane of xy—introduced into any horizontal plane depends only on the vertical displacement w. In particular, the slope of the surface depends only on w_0, the value of the vertical displacement when $z=0$; and by (3) we obviously have

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* Chapter ix. vol. i. of Love's 'Treatise on.... Elasticity' may also be usefully consulted.

$$w_0=\{(1-\eta)/(2\pi n)\}\iint(p/r)d\omega. \quad . \quad . \quad . \quad (4)$$

In (4) r is the distance between the element $d\omega$ and the point x_0, y_0 on the surface to which w_0 refers.

Relation between Pressure and Gravitational Effects.

§ 6. If we suppose t the thickness, ρ the density of the material loading the surface, its gravitational forces are derived from the potential

$$V=\gamma\iint(t\rho/r)d\omega, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where γ is the attraction between two unit masses at unit distance.

The pressure exerted by the load is

$$p=g\rho t,$$

where g is gravity at the surface. Here we may regard g as the undisturbed value prior to the application of the load, as the alteration in the vertical component is negligible for our present object. Thus

$$V=(\gamma/g)\iint(p/r)d\omega.$$

Comparing this with (4), we find for the surface value V_0 of V the simple relation

$$V_0=2\pi n\gamma w_0/\{g(1-\eta)\}. \quad . \quad . \quad . \quad . \quad (6)$$

This holds true of V_0 and w_0 all over the surface, and so applies likewise to their differential coefficients with respect to x_0 and y_0—so far at least as concerns points outside the loaded area.

The direction-cosines of the normal to the surface after the application of the load are, to a first approximation,

$$\frac{dw_0}{dx_0}, \quad \frac{dw_0}{dy_0}, \quad 1;$$

so that the slope at the point x_0, y_0 is given by

$$\psi_1=\left\{\left(\frac{dw_0}{dx_0}\right)^2+\left(\frac{dw_0}{dy_0}\right)^2\right\}^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad (7)$$

Again, to a first approximation the presence of the loading-material has altered the direction-cosines of the line of action of gravity from

$$0,\ 0,\ 1 \text{ to } \frac{1}{g}\frac{d\mathrm{V}_0}{dx_0},\quad \frac{1}{g}\frac{d\mathrm{V}_0}{dy_0},\quad 1.$$

Thus gravity has become inclined to the vertical at the angle

$$\psi_2=\frac{1}{g}\left\{\left(\frac{d\mathrm{V}_0}{dx_0}\right)^2+\left(\frac{d\mathrm{V}_0}{dy_0}\right)^2\right\}^{\frac{1}{2}}. \quad . \quad . \quad . \quad (8)$$

Employing (6) in (7) and (8), we obtain the elegant relation

$$\psi_1/\psi_2=(1-\eta)g^2/(2\pi n\gamma). \quad . \quad . \quad . \quad . \quad (9)$$

The spirit-level measures $\psi_1+\psi_2$, which always exceeds the true change of level ψ_1.

Since

$$\frac{dw_0}{dx_0}\Big/\frac{dw_0}{dy_0}=\frac{d\mathrm{V}_0}{dx_0}\Big/\frac{d\mathrm{V}_0}{dy_0}, \quad . \quad . \quad . \quad . \quad (10)$$

the final directions of gravity and of the normal to the surface lie in the same vertical plane (*i.e.* plane through z). This result may facilitate experimental investigations, as a rough idea of the direction of the resultant attraction of the loading-material will generally be obtainable by eye. The possible influence of want of symmetry in the contour of the ground, or variability of the surface-strata, must of course be borne in mind.

The relation (9), so far as I know, is new. Its discovery was due to a faint impression that a formula I had obtained for the effect of a loaded rectangle resembled something I had seen before, the something proving on investigation to be result (7) in Thomson & Tait's Nat. Phil., art. 818.

§ 7. To form an idea of the relative importance of ψ_1 and ψ_2 in the case of the earth, I have made the following selection of hypothetical values for η and n, the latter quantity being measured in grammes weight per square centimetre :—

(i.) $\eta=\cdot 25$, $n=80\times 10^7$;
(ii.) $\cdot 25$, 35×10^7;
(iii.) $\cdot 5$, 11×10^7.

According to the table in Lord Kelvin's Encyclopædia Article on Elasticity, (i.) may be regarded as representing iron or steel, (ii.) as representing brass or slate of somewhat low modulus, while (iii.) represents an incompressible material* such as seems most compatible with the hypothesis of a homogeneous earth, naturally spherical but for rotation. According to Lord Kelvin's table, the value of n in (iii.) is similar to what one should expect in rock of somewhat low elasticity.

If a be the earth's radius, $\bar{\rho}$ its mean density,

$$g/\gamma = 4\pi\bar{\rho}a/3.$$

Supposing $\bar{\rho} = 5{\cdot}5$, and $a = 64 \times 10^7$ cms., we have approximately $g\bar{\rho}a = 35 \times 10^8$ grammes weight per square centimetre.

With the above figures, I find

$$\begin{array}{lll} \text{case (i.)} & \psi_1/\psi_2 = & 35/16 = 2 \text{ roughly,} \\ \text{(ii.)} & & 10/2 = 5, \\ \text{(iii.)} & & 350/33 = 11 \text{ roughly.} \end{array}$$

The last result is likely, I think, to prove the nearest to what ordinarily occurs in practice, so that the gravitational effect may be expected to prove as a rule relatively small; still it ought not to be disregarded without due consideration of the special circumstances.

Pure Pressure Effects.

§ 8. In all cases when the largest dimension of the loaded area is small compared to its shortest distance from the point of the surface where the slope is required, a good first approximation † to the surface vertical displacement—obvious on inspection of (4)—is

$$w_0 = (1-\eta)\mathrm{P}/(2\pi n\mathrm{R}), \quad . \quad . \quad . \quad . \quad (11)$$

where R denotes the distance from the centre of mass of the total load P.

The corresponding approximation to the slope, viz.,

$$-\frac{dw_0}{d\mathrm{R}} = (1-\eta)\,\mathrm{P}/(2\pi n\mathrm{R}^2), \quad . \quad . \quad . \quad (12)$$

shows that at considerable distances from a small loaded area the slope varies approximately as the inverse square of the distance. In (11) and (12) the distribution of load is not assumed uniform.

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* See Phil. Mag. Sept. 1891, p. 250, remembering $\mathrm{E}/n = 2(1+\eta)$.

† See Todhunter & Pearson's 'History,' vol. ii. pt. 2, art. 1498.

The fact that (11) holds only when the distance of the loaded area is so large that its effect is relatively small diminishes its value in practice.

§ 9. The determination of w from (3) entails the evaluation of two integrals, neither very manageable. For points on the surface there is, however, only the single integral (4). This has been converted by Boussinesq into two alternative forms—one for points outside, the other for points inside the loaded area—which are convenient when the load, though not necessarily uniform, is distributed symmetrically round a point (see Todhunter & Pearson's 'History,' arts. 1501 and 1502). In this way the depression can be easily determined at the centre and edge of a circular depressed area for a variety of laws of loading, and the depression at other points can be expressed in terms of infinite series or elliptic integrals (see Todhunter & Pearson, *l. c.*, especially art. 1504).

The slope in these cases, at any distance from the centre of the loaded area, can be obtained in the form of infinite series or elliptic functions; but results of this kind are more apt to repel than to enlighten the unmathematical reader.

Fortunately, when the load is uniform, and the loaded area rectangular, it proves possible to express the components of slope dw/dx, dw/dy at any point of the surface in terms of ordinary Napierian logarithms. I shall accordingly devote attention almost exclusively to this case.

§ 10. Returning to (3), let x', y' be coordinates of the element $d\omega$, so that

$$d\omega = dx'dy',$$
$$r^2 = (x-x')^2 + (y-y')^2 + z^2.$$

The loading being supposed uniform, we have

$$\frac{4\pi n}{p}\frac{dw}{dx} = 2(1-\eta)\iint \frac{d}{dx}\left(\frac{1}{r}\right) dx'dy' + \iint z^2 \frac{d}{dx}\left(\frac{1}{r^3}\right) dx'dy'.$$

But

$$\frac{d}{dx}\left(\frac{1}{r}\right) = -\frac{d}{dx'}\left(\frac{1}{r}\right), \text{ and } \frac{d}{dx}\left(\frac{1}{r^3}\right) = -\frac{d}{dx'}\left(\frac{1}{r^3}\right);$$

hence

$$\frac{4\pi n}{p}\frac{dw}{dx}=2(1-\eta)\int\left(\frac{1}{r_1}-\frac{1}{r_2}\right)dy'+z^2\int\left(\frac{1}{r_1^{\,3}}-\frac{1}{r_2^{\,3}}\right)dy', \quad . \; (13)$$

where r_1 and r_2 are the inferior and superior limits of r in the integration with respect to x'.

Suppose the origin vertically over the point where the slope is to be found, or $x=y=0$, and draw the axes of x and y parallel to the sides of the loaded rectangle. Take for the coordinates of the corners of the rectangle—

$$x_1, y_1\;;\;\; x_2, y_1\;;\;\; x_2, y_2\;;\;\; x_1, y_2\;;$$

and suppose

$$x_2 > x_1, \;\; y_2 > y_1.$$

The following result is then easily obtained from (13):

$$\frac{4\pi n}{p}\left(\frac{dw}{dx}\right)_{z=z}^{x=y=0}=2(1-\eta)\log\frac{(y_2+\sqrt{y_2^{\,2}+x_1^{\,2}+z^2})\;(y_1+\sqrt{y_1^{\,2}+x_2^{\,2}+z^2})}{(y_1+\sqrt{y_1^{\,2}+x_1^{\,2}+z^2})\;(y_2+\sqrt{y_2^{\,2}+x_2^{\,2}+z^2})}$$

$$+z^2\left\{\frac{1}{x_1^{\,2}+z^2}\left(\frac{y_2}{\sqrt{y_2^{\,2}+x_1^{\,2}+z^2}}-\frac{y_1}{\sqrt{y_1^{\,2}+x_1^{\,2}+z^2}}\right)\right.$$

$$\left.+\frac{1}{x_2^{\,2}+z^2}\left(\frac{y_1}{\sqrt{y_1^{\,2}+x_2^{\,2}+z^2}}-\frac{y_2}{\sqrt{y_2^{\,2}+x_2^{\,2}+z^2}}\right)\right\}. \quad . \; . \; (14)$$

This combined with the corresponding expression for dw/dy, which can be written down by symmetry, supplies complete information as to the slope at all depths. By putting $z=0$ in (14), or by direct calculation, we get

$$\frac{dw}{dx}\text{ (at origin)}=\frac{(1-\eta)p}{2\pi n}\log\frac{(y_2+\sqrt{y_2^{\,2}+x_1^{\,2}})\;(y_1+\sqrt{y_1^{\,2}+x_2^{\,2}})}{(y_1+\sqrt{y_1^{\,2}+x_1^{\,2}})\;(y_2+\sqrt{y_2^{\,2}+x_2^{\,2}})}, \quad . \; . \; (15)$$

where y_1, y_2 must be treated algebraically. Thus ABCD representing the loaded rectangle, DM, CN perpendiculars on Ox, we have, in the case shown in fig. 1:—

$$\frac{dw}{dx}\text{ (at O)}=\frac{(1-\eta)p}{2\pi n}\log\frac{(\mathrm{OD}+\mathrm{DM})\,(\mathrm{OB}+\mathrm{BN})}{(\mathrm{OA}+\mathrm{AM})\,(\mathrm{OC}+\mathrm{CN})}; \quad . \; (16a)$$

in the case shown in fig. 2 :—

$$\frac{dw}{dx}(\text{at O})=\frac{(1-\eta)p}{2\pi n}\log\frac{(\text{OD}+\text{DM})(\text{OB}-\text{BN})}{(\text{OA}-\text{AM})\,(\text{OC}+\text{CN})}. \quad (16b)^*$$

Fig. 1.

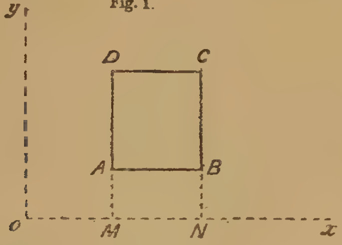


Fig. 2.

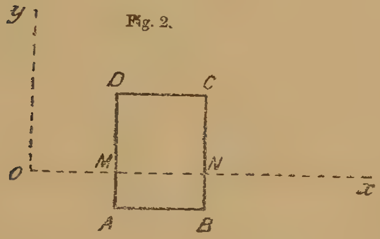
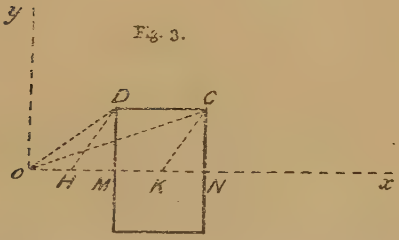


Fig. 3.



In the case of symmetry shown in fig. 3, where the centre of the rectangle lies on Ox,

$$(\text{OD}+\text{DM})/(\text{OA}-\text{AM})=(\text{OD}+\text{DM})^2/\text{OM}^2, \ \&\text{c.},$$

and (16 b) reduces to

$$\frac{dw}{dx}(\text{at O})=\frac{(1-\eta)p}{\pi n}\log\left(\frac{\text{OD}+\text{DM}}{\text{OC}+\text{CN}}\cdot\frac{\text{ON}}{\text{OM}}\right). \quad . \quad . \quad (17)$$

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* An equivalent but somewhat longer form for the logarithm (leading directly, however, to [17]) is given in eqn. (7) art. 818 of Thomson and Tait's 'Natural Philosophy.'

In this last case of course $dw/dy=0$, and Ox is the line of greatest slope at O.

If in fig. 3 we draw DH and CK, bisecting the angles ODM, OCN, we easily throw (17) into the elegant form

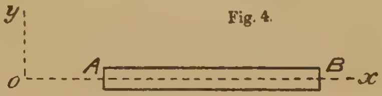
$$\frac{dw}{dx}\,(\text{at O})=\frac{1-\eta}{\pi n}\,p\log\left(\frac{\text{NK}}{\text{MH}}\right). \quad . \quad . \quad . \quad (18)$$

It will frequently be possible to divide nearly the whole of a loaded area, not itself rectangular, into a small number of rectangles, so that the results obtained above could doubtless be utilized for obtaining approximate values of the slope in many cases where the loaded area is not rectangular.

Subcases when one Dimension of Loaded Rectangle small.

§ 11. In fig. 4, AB represents an elongated loaded area symmetrical about Ox. If we suppose the breadth $2b$ small

Fig. 4.



compared to the distance $\text{OA}=c$, and denote the length AB by $2a$, we easily deduce from (17) as a first approximation

$$\frac{dw}{dx}\,(\text{at O})=(1-\eta)\text{P}\div 2\pi n\text{OA}\,.\,\text{OB}, \quad . \quad . \quad . \quad (19)$$

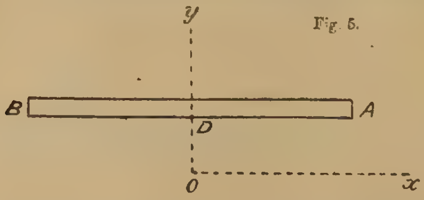
where $\text{P}\equiv p\,.\,2a\times 2b$ is the total load over the area.

If, further, c be small compared to $2a$, we have

$$\frac{dw}{dx}\,(\text{at O})=(1-\eta)pb/\pi nc=(1-\eta)\text{P}\div 4\pi nac. \quad . \quad (20)$$

When (20) applies, the slope along the axis of symmetry varies inversely as the shortest distance from the loaded area.

Fig. 5.



§ 12. In fig. 5, AB represents an elongated area perpendicular to the axis of symmetry Oy.

Supposing first that OD$(=c)$ and the breadth $2b$ are comparable, but both small compared to the length $2a$, we easily find from the formula corresponding to (17)

$$\frac{dw}{dy}\,(\text{at O})=\frac{(1-\eta)p}{\pi n}\log\frac{c+2b}{c}. \quad . \quad . \quad . \quad (21)$$

If, further, the breadth be small compared to the shortest distance from O, we reduce (21) to

$$\frac{dw}{dy}\,(\text{at O})=2(1-\eta)pb/\pi nc=(1-\eta)\mathrm{P}/2\pi nac, \quad . \quad (22)$$

where P denotes as before the total load. Comparing (20) and (22) we observe that for equal distances c, the position of the loaded area in fig. 5 is twice as effective as the position in fig. 4.

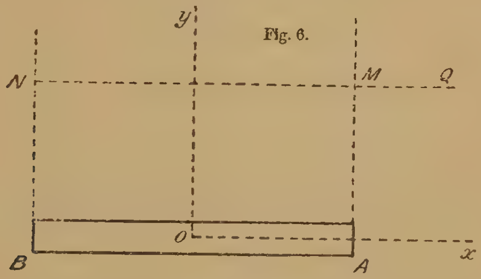


Fig. 6.

§ 13. In fig. 6 the elongated loaded area has its centre at the origin of coordinates, and the axes of Ox and Oy are along its length $2a$ and breadth $2b$ respectively. The slope is required at a point Q(x, y) whose shortest distance from the area is considerable compared to b.

Draw through Q a parallel to AB cutting in M and N the lines AM and BN drawn perpendicular to AB.

From (15) and the corresponding equation we have as first approximations to the components of slope

$$-\frac{dw}{dx}=\frac{(1-\eta)pb}{\pi n}\left(\frac{1}{\mathrm{QA}}-\frac{1}{\mathrm{QB}}\right), \quad . \quad . \quad . \quad (23)$$

$$-\frac{dw}{dy}=\frac{(1-\eta)pb\mathrm{AM}}{\pi n}\left\{\frac{1}{\mathrm{QA(QA+QM)}}-\frac{1}{\mathrm{QB(QB+QN)}}\right\} \quad (24)$$

In (24), QM and QN are to be treated algebraically, and the formula must not be applied to cases in which either QA + QM or QB + QN tends to become very small (*cf.* § 11).

Numerical Illustrations.

§ 14. Suppose in the case of symmetry, illustrated by fig. 3, that the loaded area is a square 100 metres in the side, and that OM, its shortest distance from O, is 1 metre.

Suppose the load to arise from a sheet of water 1 cm. deep, or that

$$p=1 \text{ gramme wt.}$$

As in case (iii.) of § 7, let us put

$$\eta=\cdot 5,\ n=11\times 10^7 \text{ grammes wt. per sq. cm.}$$

Then we have approximately, in absolute measure,

$$\frac{dw}{dx}=(2\pi\times 11\times 10^7)^{-1}\log_e(10100/163)\,;$$

or, as unit angle $=206\times 10^3$ seconds of arc approximately,

$$\text{slope at O}=0''\cdot 0012 \text{ approximately.} \quad . \quad . \quad (25)$$

The result would be the same if the side and the least distance of the loaded square were altered in the same proportion, *e. g.* if the side were altered to 1000 and the shortest distance to 10 metres.

The slope increases directly as the load. It would, however, require an abnormally large differential rainfall or evaporation to appreciably influence by direct pressure a level inside a building situated on strata similar to the material of our calculation.

§ 15. The differential effect of barometric pressure during the approach or retirement of a deep cyclonic depression would appear a more probable disturbing cause. We might very easily have a mean differential excess or diminution of pressure of 1 or 2 cm. of *mercury* over an area whose greatest dimension was very large compared to the shortest distance from the observing station, and consequently effects 10 or 20 times that appearing in (25) might not unreasonably be expected in disturbed weather.

In the case of a large cyclonic area it would be desirable to apply a formula applicable to a loaded *spherical* surface, but (17) would probably give a very fair idea of the order of magnitude of the result.

§ 16. As an illustration of a different kind, suppose in fig. 5 that O is a station near a long straight portion AB of a tidal river, and that we desire the difference of slope at O at high and low tide. It will suffice to take the difference of level at high and low water as the same all along AB. Suppose this difference to be 5 metres, and assume η and n to be the same as in the last example.

Taking first $c=4\times 2b$, we get approximately from (21)

$$\frac{dw}{dy}(\text{at O})=\frac{5\times 10^2\times 7}{2\times 22\times 11\times 10^7}\log_e 10\times\log_{10}(1{\cdot}25),$$

or in seconds of arc

$$\text{slope at O}=0''{\cdot}033. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (26)$$

Taking next $c=2b$, we replace (26) by

$$\text{slope at O}=0''{\cdot}10. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (27)$$

If, for instance, the river be 100 yards broad, the first station is 400, the second 100 yards from the bank.

§ 17. As rivers are seldom straight, I have supplemented the above by calculating the slope at the centre of a semicircular channel of width $2b$, supposed small compared to the radius R. For a difference h in the level of high and low water, I find as a first approximation

$$\text{difference of slope at centre of semicircle}=2(1-\eta)gbh/\pi n\text{R}. \qquad (28)$$

To aid the imagination, the river may be supposed to enter and leave the semicircle by straight channels forming continuations of the limiting diameter, so that the semicircular portion alone need be considered.

It will be noticed that (22) and (28) are identical if

$$c=\text{R, and } p=gh.$$

In other words, the semicircular tidal river has exactly the same influence on the slope of the station at its centre as it would have if the channel were straight throughout and came within the same distance of the station.

§ 18. The results of the last two paragraphs point to changes of water-level in tidal rivers and estuaries as more likely to appreciably affect the level of neighbouring observatories than any probable differential peculiarities of rainfall or evaporation. In making this observation I exclude of course the direct action of water on the foundations of the building.

In all cases similar to those treated in §§ 14–17 the direct gravitational action of the load must be taken into account to obtain the full result. For instance, in the case of an anti-cyclone, the horizontal attraction of the surplus air must be considered as well as the excess of pressure over the area covered by the anti-cyclone.

Pressure Effects below the Surface.

§ 19. As the foundations of most buildings are below the ground-level, the slope at some little depth possesses considerable interest. The general formula (14) for the depression at any depth due to a loaded rectangle, though easily evaluated for specified numerical values of z, x, &c., is somewhat complicated. Its general character will be sufficiently comprehended from the results in the case of symmetry, when

$$y_2 = -y_1 = b.$$

Putting $$x_1 = c,\ x_2 = c + 2a,$$

we then convert (14) into

$$\frac{4\pi n}{p}\left(\frac{dw}{dx}\right)_{z=z}^{x=y=0}$$

$$= 2(1-\eta)\log\frac{(b+\sqrt{b^2+c^2+z^2})\,(-b+\sqrt{b^2+(c+2a)^2+z^2})}{(-b+\sqrt{b^2+c^2+z^2})\,(b+\sqrt{b^2+(c+2a)^2+z^2})}$$

$$+2bz^2\{(c^2+z^2)^{-1}(b^2+c^2+z^2)^{-\frac{1}{2}}$$

$$-((c+2a)^2+z^2)^{-1}(b^2+(c+2a)^2+z^2)^{-\frac{1}{2}}\}. \quad . \quad . \quad (29)$$

So long as z/c is small the right-hand side of (29) can be expanded in a rapidly converging series of the form

$$A + B(z/c)^2 + \ldots,$$

where A and B are independent of z.

There is no tendency in B/A to become very large for finite values of a/c and b/c.

When we neglect $B(z/c)^2$ &c., we simply get the slope at the surface. We thus see that at depths small compared to the shortest distance of the loaded area the slope is nearly the same as at the surface itself.

The value of B is easily obtained in special cases. As an example, take the sub-case of fig. 4, in which c/a and b/c are both supposed small. We then get for the slope

$$\left(\frac{dw}{dx}\right)_{z=z}^{x=y=0} = \frac{pb}{\pi nc}\left(1-\eta+\tfrac{1}{2}\eta\frac{z^2}{c^2}\right). \quad . \quad . \quad . \quad (30)$$

In (30) constant terms of the order $(b/c)^3$ are omitted, though possibly more important than the variable term.

Again, in the sub-case of fig. 5, when b/c and c/a are supposed both small, we find

$$\left(\frac{dw}{dy}\right)_{z=z}^{x=y=0} = \frac{2pb}{\pi nc}\left(1-\eta+\eta\frac{z^2}{c^2}\right). \quad . \quad . \quad . \quad (31)$$

In both these instances the slope *increases* with the depth. The formulæ hold only so long as z/c is small, so that the phenomenon is rather of theoretical than practical importance. Though somewhat opposed to *à priori* conceptions, this result would not appear exceptional. Thus, take the case of an isolated load $pd\omega$ at a point P on the surface, and consider the vertical displacement at a point Q at depth z. Join QP$=r$, and draw QM$=x'$ perpendicular on the vertical PM. Then denoting the angle QPM by α, we have at Q

$$w=(pd\omega/4\pi nr)\{2(1-\eta)+\cos^2\alpha\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)^*$$
$$=(pd\omega/4\pi nx')\{2(1-\eta)(1+z^2/x'^2)^{-\frac{1}{2}}+(z/x')^2(1+z^2/x'^2)^{-\frac{3}{2}}\}.$$

Thus when z/x' is small, we find, neglecting powers of z/x' above the second,

$$w=(pd\omega/2\pi nx')\{1-\eta+\tfrac{1}{2}\eta(z/x')^2\}; \quad . \quad . \quad . \quad (33)$$

whence

$$-\frac{dw}{dx'}=(pd\omega/2\pi nx'^2)\{1-\eta+\tfrac{3}{2}\eta(z/x')^2\}. \quad . \quad . \quad (34)$$

Here the slope $-dw/dx'$ increases at first with the depth as in the case of (30) and (31).

§ 20. When the depth is of the same order of magnitude as the horizontal distance of the nearest point of the loaded area, individual cases of (14) or (29) require separate consideration.

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[*] Todhunter & Pearson's 'History,' vol. ii. eqn. (xxiv.) of art. 1497.

When the depth becomes large compared to the horizontal distance of the remotest point of the loaded area, we easily find from (14) as a first approximation

$$\left(\frac{dw}{dx}\right)_{z=z}^{x=y=0}=\frac{(5-2\eta)p}{8\pi n}\frac{(y_2-y_1)(x_2{}^2-x_1{}^2)}{z^3}, \quad . \quad (35)$$

showing that the slope now diminishes as the inverse cube of the depth.

If P denote the total load, $\bar{x}, \bar{y}$ the coordinates of the C.G. of the loaded area, we have at once from (35)

$$\left(\frac{dw}{dx}\right)_{z=z}^{x=y=0}=(5-2\eta)\mathrm{P}\bar{x}/(4\pi nz^3), \quad . \quad . \quad . \quad (36)$$

and by symmetry

$$\left(\frac{dw}{dy}\right)_{z=z}^{x=y=0}=(5-2\eta)\mathrm{P}\bar{y}/(4\pi nz^3). \quad . \quad . \quad . \quad (37)$$

The line of greatest slope is thus in the vertical plane which contains the C.G. of the loaded area, and if R be the horizontal distance of the C.G., the slope is given by

$$(dw/d\mathrm{R})_{z=z}=(5-2\eta)\mathrm{PR}/(4\pi nz^3). \quad . \quad . \quad . \quad (38)$$

The conditions assumed in (35) are practically tantamount to those of the elementary loaded area, and (38) can in fact be deduced from (32) by supposing α small.

*Luni-Solar Effects.**

§ 21. Another possible cause affecting the indications of pendulums and spirit-levels is the gravitational action of the heavenly bodies, especially the sun and moon. If we regard the earth as a sphere of mass E and radius a, and suppose the moon's mass to be M and its distance from the earth R, there exists in the earth a system of bodily forces of which the principal come from a potential

$$g(\mathrm{M}/\mathrm{E})(a/\mathrm{R})^3(r^2/a)(3\cos^2\theta-1)/2, \quad . \quad . \quad . \quad (39)$$

where g is "gravity" at the earth's surface, neglecting "centrifugal force." The moon is supposed to lie in the line

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* Strictly the problem is a dynamical one; as yet only an "equilibrium" solution is available.

$\theta=0$, the earth's centre being origin, and r, θ ordinary polar coordinates. As explained in art. 812 of Thomson and Tait's 'Natural Philosophy,' there results at the earth's surface a component force radially outwards

$$g(\mathrm{M}/\mathrm{E})(a/\mathrm{R})^3(3\cos^2\theta-1),$$

and a component along the tangent

$$\mathrm{T}'=3g(\mathrm{M}/\mathrm{E})(a/\mathrm{R})^3\sin\theta\cos\theta, \quad . \quad . \quad . \quad . \quad (40)$$

directed towards the point under the moon ($\theta=0$).

Both components being small compared to g, the direction of gravity is, owing to the direct attraction alone, deflected through the angle

$$\delta\psi'=\tan^{-1}\{3(\mathrm{M}/\mathrm{E})(a/\mathrm{R})^3\sin\theta\cos\theta\} \quad . \quad . \quad (41)$$

from the vertical. The angle being very small may be replaced by its tangent.

Thomson and Tait suppose

$$(\mathrm{M}/\mathrm{E})(a/\mathrm{R})^3=10^{-5}/182, \quad . \quad . \quad . \quad . \quad (42)$$

and thence draw the following conclusion:—"the plummet is deflected towards the point of the horizon under either moon ($\theta=0$) or antimoon ($\theta=\pi$), by an amount which reaches its maximum value ... $0''\cdot017$ when the altitude is 45°." They add—"The corresponding effects of solar influence are of nearly half these amounts." According to this conclusion direct luni-solar influence should make itself felt in any system of pendulum or spirit-level observations in which the accuracy is of the order $0''\cdot02$.

§ 22. The data on which the above calculation is based are pretty accurately known, which constitutes a reason for treating the direct effect by itself. It must, however, be borne in mind that the luni-solar influence is not confined to the pendulum bob, but extends to the material of the earth itself. Consequently the result calculated by Lord Kelvin and Prof. Tait is part only of a composite effect, which there is no very obvious way of analysing in practice into its components.

In the actual earth the most obvious consequence of luni-solar action is the ocean tides, and, as we saw in § 4, any station near the sea-shore has its apparent level affected by these in two distinct ways. Even at an inland station ocean tides must exert some influence, though presumably it is very small.

In addition, however, to ocean tides there must be tides in the earth's mass, whether solid throughout or not, and it is to these I shall now call attention.

The potential term (39) is only one of a series. The numerical values of the coefficients diminish rapidly as the order of the harmonic increases; still it is desirable not wholly to neglect the higher harmonics, if only to make sure that the comparative smallness of the disturbing forces answering to them is not compensated in any instance by great effectiveness. I shall thus consider in the first place the results of the general problem when the degree of the harmonic appearing in the disturbing forces is unrestricted, making use of the results contained in a paper communicated to the Cambridge Philosophical Society* in 1887.

§ 23. Before entering, however, on this investigation, it is desirable to consider briefly the relation between the results of theory and the phenomena we may expect to encounter in direct observation.

Surface-points on the undisturbed surface, regarded as spherical, transform into surface-points on the strained surface; thus a very small surface-area, *e.g.* a square decimetre, may be regarded as a tangent plane in both conditions.

Suppose, now, this area to have rigidly attached to it a spirit-level, consisting of part of a circular arc filled with liquid and with a minute air-bubble. In the undisturbed condition suppose the bubble exactly at the central division O of the arc, while in the disturbed condition it is at an angular interval $\delta\theta$ from O. This an observer would naturally attribute to a change $\delta\theta$ of level. The true interpretation is that in the disturbed condition the resultant of the forces at the surface makes with the normal the angle $\delta\theta$. In a rigid earth $\delta\theta$ would be the angle of Thomson and Tait's calculation; but in an elastic earth allowance must be made for the fact that the attraction of the disturbed earth is not along the normal.

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* See the Society's Transactions, vol. xiv. p. 278.

The effect on astronomical observations is still more complicated. Thus let an observer take the altitude of a star in the same vertical plane as the moon, using a mercury surface for his horizontal plane. The observed altitude will differ from the theoretical—*i. e.* the true altitude if the disturbing influence were absent—by an amount equal to the angle between the disturbed and undisturbed mercury surfaces. This is the algebraical sum of the inclination of the resultant gravitational force to the radius-vector in the disturbed condition and of the inclination of this radius-vector to its undisturbed position.

This explanation will show what the quantities are of which we require to know the theoretical values.

§ 24. To return to the problem. The earth is treated as truly spherical when undisturbed, "centrifugal force" being neglected, and as posssesed when disturbed of uniform density ρ, and of uniform isotropic elastic qualities throughout, determined by the elastic constants m, n.

The assumption of natural sphericity and the neglect of the centrifugal force answer merely to the neglect of small quantities of the second order of magnitude relative to those of the first; the other assumptions have been discussed in § 2. In our ultimate applications the material will be supposed incompressible, i. e. $n/m=0$, but it is undesirable to introduce unnecessary limitations in the mathematical results themselves. Further, absolutely incompressible material is merely a mathematical fiction, so it is desirable to have the means ready to hand to apply a correction to mathematical results based on such an hypothesis.

Supposing the typical term in the potential of the disturbing forces to be

$$r^i V_i' \sigma_i, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

where σ_i is a known surface-harmonic of degree i, and V_i' a given numerical magnitude, we easily see that the equation to the strained surface will take the form

$$r=a+\sum a_i \sigma_i. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

At the present stage all we know is that a_i is small compared to a, the mean radius of the strained surface.

The bodily forces consist in part of the disturbing forces, but mainly of the self-gravitational action of the "earth." The complete value of the potential V is given by

$$V = -\tfrac{1}{2} g a^{-1} r^2 + \Sigma\, 3g (r/a)^i a_i \sigma_i / (2i+1) + \Sigma\, r^i V_i' \sigma_i \text{[*]}. \quad . \quad (45)$$

In ordinary circumstances we are supposed to be given the unstrained surface, with full information as to the force system, and it is customary to regard the surface equations as applying to the unstrained surface. In the present instance—and I daresay as a rule in practice—the forces depend to some extent on the disturbed form of the body. It is thus convenient, to say the least of it, to suppose that the surface equations apply in the present instance to the disturbed surface. This implies nothing more serious than the replacing the ordinary definition of strain, viz.

(final length—initial length)/(initial length),

by

(final length—initial length)/(final length).

The two definitions are equivalent so long as it is justifiable to apply the mathematical theory, which assumes the square of a strain negligible[†].

§ 25. The problem whose results I am about to use was more general than the one at present before us, inasmuch as the surface was not assumed to be naturally spherical. The notation employed in its solution was also somewhat different, the potential being given in the form

$$V = -\tfrac{1}{2} g a^{-1} r^2 + \Sigma\, a_i V_i \sigma_i r^i. \quad . \quad . \quad . \quad . \quad (46)$$

Thus in utilizing the results we must put

$$V_i = 3g a^{-i}/(2i+1) + V_i'/a_i. \quad . \quad . \quad . \quad . \quad (47)$$

In the general problem V_i was unrestricted, but I contented myself with giving the two arbitrary constants $a_i Y_i$, $a_i Z_i$ explicitly in terms of $a_i V_i$ and $g a^{-i} a_i$. The expressions for the displacements freed from arbitrary constants were given (*l. c.* equations (13) to (15), pp. 280, 281) only for the case when

$$V_i = 3g a^{-i}/(2i+1),$$

or when V_i' in (47) is zero.

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[*] See Prof. G. H. Darwin, Phil. Trans. 1879, Part I.

[†] See Phil. Mag. Sept. 1891, pp. 246–7.

It is easy, however, to add the terms containing V_i'. For if in the equations (11) and (12) of p. 280, *l. c.*, we substitute for V_i the right-hand side of (47) and multiply up by $a_i\sigma_i$, we notice that $V_i'\sigma_i$ appears with the same coefficients as V_i possessed in the earlier equations (32) and (33) (*l. c.* p. 264), which determined the unknowns Y_i and Z_i—treated in that instance as each the combined product of surface harmonic and arbitrary coefficient—for a perfect sphere acted on by bodily forces. Thus for terms in V_i' in the displacements, we have only to take the results (36), (37), and (38), *l. c.* pp. 264–265, and in them replace V_i by $V_i'\sigma_i$ and S_i by zero.

Doing so, we find for the components of displacement, measured respectively in the directions of the fundamental polar elements dr, $r\,d\theta$, $r\sin\theta\,\delta\phi$, the following results :—

$$u = \frac{g\rho r}{10a(m+n)}\left\{r^2 - \frac{a^2(5m+n)}{3m-n}\right\}$$

$$+g\rho\Sigma\frac{a_i\sigma_i}{2n(2i+1)D_i}[r^{i+1}a^{-i}\{10m^2i(i^2-1)$$

$$-mn(4i^4+4i^3+34i^2+29i+10)+n^2(8i^3+8i^2+13i-2)\}$$

$$-r^{i-1}a^{-i+2}i\{10m^2i(i+2)-mn(4i^3+4i^2-2i+1)-n^2(4i-3)\}]$$

$$+\frac{5(m+n)}{2n}\Sigma\frac{\rho i V_i'\sigma_i}{D_i}\left[a^2r^{i-1}i\frac{m(i+2)-n}{i-1} - r^{i+1}\{m(i+1)-n\}\right], \quad (48)$$

$$v = \frac{d}{d\theta}\Sigma\Psi_i, \quad w = \frac{1}{\sin\theta}\frac{d}{d\phi}\Sigma\Psi_i; \quad . \quad . \quad . \quad (49)$$

where

$$D_i = 5(m+n)\{m(2i^2+4i+3)-n(2i+1)\}, \quad . \quad . \quad (50)$$

$$\Psi_i = \frac{g\rho a_i\sigma_i}{2n(2i+1)D_i}[r^{i+1}a^{-i}\{10m^2(i-1)(i+3)$$

$$-mn(4i^3+12i^2-6i+17)-n^2(8i^2-17)\}$$

$$-r^{i-1}a^{-i+2}\{10m^2i(i+2)-mn(4i^3+4i^2-2i+1)-n^2(4i-3)\}]$$

$$+\frac{5(m+n)\rho V_i'\sigma_i}{2nD_i}\left[a^2r^{i-1}\frac{i\{m(i+2)-n\}}{i-1} - r^{i+1}\{m(i+3)-n\}\right]. \quad (51)$$

§ 26. Before utilizing these results we must determine a_i in terms of V_i', which is easily done as follows :—The surface being supposed originally spherical, the terms $\Sigma a_i\sigma^i$ in (44) arose solely from the action of the disturbing forces, and so must be identical with the variable terms in the surface-value of u.

Thus writing $a+\Sigma a_i\sigma_i$ for r in the principal terms in (48), and a for r in the subsidiary, then equating the separate harmonic terms to the corresponding ones in $\Sigma a_i\sigma_i$ and reducing, we find

$$a_i=\frac{\rho a^{i+1}\mathrm{V}_i'i\{(2i+1)m-n\}\div\{2(i-1)n((2i^2+4i+3)m-(2i+1)n)\}}{1+\dfrac{g\rho a}{n}\dfrac{15i(2i+1)m^2-(8i^3+6i^2-2i-9)mn+(4i^3-2i^2-3i-3)n^2}{5(2i+1)(3m-n)\{(2i^2+4i+3)m-(2i+1)n\}}}. \quad (52)$$

If the self-gravitation were negligible, the denominator in (52) would become unity, the numerator remaining unchanged. Thus self-gravitation reduces the change of form produced by the disturbing forces depending on the harmonics of degree i in the ratio

$$1:1+\frac{g\rho a}{n}\frac{15i(2i+1)m^2-(8i^3+6i^2-2i-9)mn+(4i^3-2i^2-3i-3)n^2}{5(2i+1)(3m-n)\{(2i^2+4i+3)m-(2i+1)n\}} \quad (53)$$

If
$$n/m=0,$$

or the material be incompressible, (52) reduces to

$$a_i=\frac{\rho a^{i+1}\mathrm{V}_i'i(2i+1)/\{2(i-1)(2i^2+4i+3)n\}}{1+(g\rho a/n)i/(2i^2+4i+3)}, \quad . \quad . \quad (54)$$

and the ratio (53) becomes

$$1:1+(g\rho a/n)i/(2i^2+4i+3). \quad . \quad . \quad . \quad (55)$$

When $i=2$, (52) becomes

$$a_2=\frac{\rho a^3\mathrm{V}_2'(5m-n)/\{n(19m-5n)\}}{1+3(g\rho a/n)(10m^2-5mn+n^2)/\{5(3m-n)(19m-5n)\}}. \quad . \quad (56)$$

A result equivalent to (56), with the notation

$$\lambda=m-n, \qquad \mu=n,$$

was given by Prof. Karl Pearson, in Todhunter and Pearson's 'History,' vol. ii. part 2, p. 425. An obvious misprint of 4μ for 14μ occurs, however, in the denominator of his formula.

§ 27. We may safely assume $m-n$ positive, so the numerator in (52) has clearly the same sign as V_i'; also for a given value of V_i' it diminishes as i increases. Thus so long as the denominator in (52) exceeds unity there is no risk lest the relative smallness of the forces proceeding from any higher harmonic may be compensated in any way. It is obvious, however, that the coefficient of $g\rho a/n$ in the denominator can be made negative by taking i large enough, for ordinary values of n/m. For instance, if $n/m=1/2$,—the hypothesis of uniconstant isotropy,—the coefficient of $g\rho a/n$ is negative when i exceeds 9, and with i infinite the denominator as a whole would vanish and change sign as n passed through the value $6g\rho a/100$.

If we take as before

$$g\rho a=35\times10^8 \text{ grammes wt. per sq. cm.},$$

this critical value of n has the very ordinary value 21×10^7 grammes wt. per sq. cm.

Thus if such a value as $m/n=2$ were admissible the contingency of a_i/V_i' becoming enormously large for a high value of i would be quite a possible one. Unless, however, as I have previously pointed out, n/m be very small, the term in u, independent of the angular coordinates, would in a body of the earth's mass be enormously greater than is consistent with the mathematical theory of elasticity. Therefore, so long as the present calculation is justifiable, the denominator in the value of a_i/V_i' can differ but little from that occurring in (54), and we are thus thoroughly justified in neglecting all the higher harmonic terms in the potential relative to the term containing the second harmonic.

§ 28. As a small departure of n/m from 0 would exercise but little influence on numerical values, it will be best, as we are dealing with data so uncertain, to neglect n/m altogether.

Thus, putting

$$i=2, \quad \sigma_2=P_2, \quad n/m=0,$$

$$V_2'=g(M/E)(a^2/R^3), \quad . \quad . \quad . \quad . \quad . \qquad (57)$$

we have for the displaced surface

$$r=a+a_2P_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (58)$$

where

$$a_2/a=\frac{5}{19}(g\rho a/n)(\mathrm{M}/\mathrm{E})(a/\mathrm{R})^3\div\left\{1+\frac{2}{19}(g\rho a/n)\right\}. \quad (59)$$

An equivalent result is given in Thomson and Tait's 'Natural Philosophy,' art. 840. The result will also be found, along with that answering to $i=2$, $m=2n$, in Mr. Love's 'Treatise on Elasticity,' vol. i. pp. 302, 303.

The corresponding surface displacements are

$$u_a=\frac{5}{19}a\mathrm{P}_2(g\rho a/n)(\mathrm{M}/\mathrm{E})(a/\mathrm{R})^3/\{1+2g\rho a/19n\}, \quad (60)^*$$

$$v_a=-\frac{9}{38}a\sin\theta\cos\theta(g\rho an)(\mathrm{M}/\mathrm{E})(a\mathrm{R})^3/\{1+2g\rho a/19n\}. \quad (61)^*$$

The term in u independent of the angular coordinates absolutely vanishes for $n/m=0$, and both components of the surface displacement, and so the resultant displacement itself, are reduced owing to the self-gravitation in the common ratio

$$1:1+2g\rho a/(19n). \quad . \quad . \quad . \quad . \quad (62)^*$$

The angle through which the radius-vector is rotated from its undisturbed position, in the direction away from $\theta=0$, is equal to v_a/a, and so is known from (61). As v_a/a is negative for all values of θ between 0 and $\pi/2$, this rotation is really *towards* the moon at every point of the illuminated hemisphere.

§ 29. We next require to find the inclination of the resultant force to the radius-vector over the surface.

Employing (57) and (59) in (45) we find for the complete value of the potential

$$\mathrm{V}=-\tfrac{1}{2}g(r^2/a)[1-2\mathrm{P}_2(\mathrm{M}/\mathrm{E})(a/\mathrm{R})^3(1+5g\rho a/19n)$$
$$\div(1+2g\rho a/19n)]. \quad . \quad (63)$$

The component forces along and perpendicular to the radius-vector are

$$\overline{\mathrm{R}}=\frac{d\mathrm{V}}{dr}, \qquad \overline{\Theta}=\frac{1}{r}\frac{d\mathrm{V}}{d\theta}.$$

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* The material being as here incompressible, it may be proved that for any value of i in (45) the *displacements* are everywhere the same as in a *sphere* of radius a, over whose surface act purely normal tractions equal to $\rho a^i\sigma_i\mathrm{V}_i'\div\{1+(g\rho a/n)i/(2i^2+4i+3)\}$.

Thus at a point on the surface the principal terms, which alone we require, are

$$\overline{\mathrm{R}}=-g,\ \overline{\Theta}=-3g\sin\theta\cos\theta(\mathrm{M}/\mathrm{E})(a/\mathrm{R})^3(1+5g\rho a/19n) \div(1+2g\rho a/19n), \quad . \quad (64)$$

and the inclination of the resultant force to the radius-vector is to a first approximation

$$\delta\chi=\overline{\Theta}/\overline{\mathrm{R}}=3\sin\theta\cos\theta(\mathrm{M}/\mathrm{E})(a/\mathrm{R})^3(1+5g\rho a/19n) \div(1+2g\rho a/19n). \quad . \quad (65)$$

For the apparent change of altitude, $\delta\alpha$, in a star we have, as already explained in § 23,

$$\delta\alpha=\delta\chi+v_a/a=3\sin\theta\cos\theta(\mathrm{M}/\mathrm{E})(a/\mathrm{R})^3\left\{1+\frac{7}{38}(g\rho a/n)\right\} \div(1+2g\rho a/19n). \quad . \quad (66)$$

§ 30. For the apparent change of level $\delta\psi$, we require the inclination of the resultant force to the normal. To obtain this we may employ the result (65) in conjunction with the inclination $\delta\psi_1$ of the normal to the radius-vector, the latter being given to a first approximation by

$$\delta\psi_1=\left(-\frac{1}{r}\frac{dr}{d\theta}\right)_{r=a}=\frac{15}{19}\sin\theta\cos\theta(g\rho a/n)(\mathrm{M}/\mathrm{E})(a/\mathrm{R})^3 \div(1+2g\rho a/19n). \quad (67)$$

Thus we have finally

$$\begin{aligned}\delta\psi&=\delta\chi-\delta\psi_1\\&=3\sin\theta\cos\theta(\mathrm{M}/\mathrm{E})(a/\mathrm{R})^3\div(1+2g\rho a/19n). \quad (68)\end{aligned}$$

This result can also be got by noticing that

$$\begin{aligned}\delta\psi&=\tan^{-1}(\mathrm{T}/g),\\&=\mathrm{T}/g, \text{ to a first approximation,}\end{aligned}$$

where

$$\begin{aligned}\mathrm{T}&\equiv\overline{\mathrm{R}}\sin\delta\psi_1-\overline{\Theta}\cos\delta\psi_1\\&=3g\sin\theta\cos\theta(\mathrm{M}/\mathrm{E})(a\mathrm{R})^3\div(1+2g\rho a/19n). \quad . \quad (69)\end{aligned}$$

is the tangential component of the surface-force.

Comparing (68) with (41) we have

$$\delta\psi : \delta\psi' : : 1 : 1+2g\rho a/19n, \quad . \quad . \quad . \quad (70)$$

or the self-gravitational forces reduce the apparent change of level, as calculated in Thomson and Tait's 'Natural Philosophy,' in precisely the same ratio as they reduce the ellipticity of the surface.

This last result might probably be deduced at once from the fact that u_a and v_a are reduced in the same proportion, but I have preferred an explicit mathematical proof.

Comparing (66) and (68), we have

$$\delta\alpha/\delta\psi = 1+\frac{7}{4}(2g\rho a/19n), \quad . \quad . \quad . \quad . \quad (71)$$

showing that the apparent change in star's altitude—the star being, it will be remembered, in the same vertical plane with the moon—is always in excess of the apparent change of level.

Numerical Estimates.

§ 31. As before, we shall take

$$a = 64 \times 10^7,$$
$$g\rho a = 35 \times 10^8 \text{ grammes wt. per sq. cm.},$$
$$(\mathrm{M/E})\,(a/\mathrm{R})^3 = 1/(182 \times 10^5).$$

We shall consider only the greatest and least values of n specified in § 7, exhibiting the results side by side; θ, it will be remembered, is measured from the line joining the centres of the earth and moon.

Numerical Results: Lunar Influence.

$n =$	80×10^7 grammes wt. per sq. cm.	11×10^7 grammes wt. per sq. cm.
$2g\rho a/19n =$	$35/76$	$700/209$
$1 : 1+2g\rho a/19n$ (approx.) $=$	$11 : 16$	$3 : 13$
$u_a =$	$28\,\frac{3\cos^2\theta-1}{2}$ cms.	$68\,\frac{3\cos^2\theta-1}{2}$ cms.
$v_a =$	$-25 \sin\theta\cos\theta$ cms.	$-61 \sin\theta\cos\theta$ cms.
Polar less equatorial radius $=$	42 cms.	102 cms.
Apparent change level, $\delta\psi =$	$0''{\cdot}012 \sin 2\theta$	$0''{\cdot}004 \sin 2\theta$
Apparent change of star's altitude, $\delta\alpha =$	$0''{\cdot}021 \sin 2\theta$	$0''{\cdot}027 \sin 2\theta$

The reduction effected by the self-gravitational forces in Thomson and Tait's estimate, $0''{\cdot}017 \sin 2\theta$, for the apparent change of level increases conspicuously as the rigidity diminishes. In fact, for the lower value of n, $\delta\psi$ would be insensible unless with an instrument recording to $\frac{1}{250}$ of a second of arc.

On the other hand, the changes in the shape of the earth and in the star's apparent altitude are very decidedly larger for the lower value of n.

Corresponding results of about half the numerical size of the above would be obtained in the case of solar influence.

Final Conclusions.

§ 32. The results obtained indicate at least the directions in which luni-solar effects may be profitably looked for. If the earth's elasticity for luni-solar influence be perfect, apparent changes of level or star's altitude will be *nil* when the moon or sun, as the case may be, is either in the zenith or on the horizon, while they will be a maximum when the altitude is 45°. If the elasticity be not perfect, a lag in the tides may be expected. As regards star's altitude, a hopeful feature is that the influence, being the same for all stars in the same vertical, should be easily separable from terrestrial refraction. The further fact that the apparent change is proportional to the cosine of the star's azimuth measured from the vertical plane containing the moon, or sun, may prove of assistance.

It would appear that luni-solar effects are not unlikely to prove of as much consequence as the direct pressure or gravitational effects of any ordinary differential meteorological action in the neighbourhood of an observatory, though not nearly so important as ocean or estuary tides for observatories situated within a few hundred yards of high-water mark.

The considerable fluctuation of the calculated luni-solar effects with the value ascribed to the earth's rigidity may lead eventually to interesting speculations as to the state of the earth's interior.

Subsidiary Remarks.

§ 33. Whilst attention has been confined to surface-pressure and luni-solar action, it is not intended to imply the non-existence of other agents capable of producing similar phenomena. The sun's direct heating effect is doubtless in some cases a most effective agent in altering the level. *A priori* one would expect a diurnal variation from this cause, most sensible at stations on rocky ground exposed to the south.

§ 34. Before quitting the subject, it is desirable to consider what light existing seismological data throw on the credibility of the hypothetical theory adopted.

It appears pretty generally believed that wave-velocities calculated from observations near and distant from the epicentre of an earthquake are usually different, and the existence of at least two widely different wave-velocities seems on some occasions well established at the distant stations. One of the two wave-velocities has been regarded (on, I think, mistaken grounds) as postulating an elasticity incredibly high for an elastic solid medium.

These phenomena are easily reconciled with the elastic solid hypothesis. When waves travel between two distant points through the interior of a sphere of large radius they may be expected to behave much as if the medium were infinite. Now in an infinite isotropic medium*, as is well known, there are two wave-velocities, v_1 and v_2, given in our previous notation by

$$v_1 = \sqrt{(m+n)/\rho} \quad v_2 = \sqrt{n/\rho}.$$

Thus, under the conditions supposed, we should expect two earthquake-waves with velocities similar to v_1 and v_2. For definiteness, suppose that the velocities are actually v_1 and v_2, and suppose them to be respectively 12·5 and 2·5 kilometres per second, this appearing a fair estimate.

Then, in absolute C.G.S. measure,

$$\sqrt{(m+n)/\rho} = 125 \times 10^4, \quad \sqrt{n/\rho} = 25 \times 10^4.$$

Taking $\rho = 5{\cdot}5$ for the earth, we have the approximate results,

$$n = m/24 = 35 \times 10^7 \text{ grammes wt. per sq. cm.}$$

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* See, for instance, Love's 'Treatise on Elasticity,' vol. i. pp. 133, 134.

For E, Young's modulus, and k, the bulk modulus (resistance to compression), we have similarly,

$$E \equiv n(3-n/m) = 10 \times 10^8 \text{ grammes wt. per sq. cm.,}$$
$$k \equiv m-n/3 \quad = 83 \times 10^8 \quad ,, \quad ,, \quad ,,$$

The rigidity and Young's modulus—the quantities from whose magnitudes our conception of a material's elasticity is usually derived—are in no ways remarkable, being much below the average magnitude observed in iron. The only abnormal feature is the enormous resistance to compression. Any one, however, who considers the enormous pressures presumably in continuous operation on the earth's deep-seated material, will appreciate the probability that it responds uncommonly little to any slight increase in pressure.

A difference between the velocities calculated at stations near and distant from the epicentre is only what we should expect. Lord Rayleigh * has shown that waves with a velocity somewhat less than $\sqrt{n/\rho}$ may be propagated through the material close to the surface of a medium bounded by an infinite plane; and a similar phenomenon may be expected in a sphere, so long at least as the distance from the epicentre is small compared to the radius. In such waves the velocity must depend mainly on the density and elastic properties of the surface material, which in general must differ largely from the corresponding quantities in the deep-seated material. Thus the velocities calculated from the observed effects must depend largely on whether the waves propagated along the surface or those propagated through the interior are the dominant ones; in other words, on whether the distance of the station from the epicentre is or is not small compared to the earth's radius.

DISCUSSION.

Prof. PERRY said he had thought of taking up the subject from an experimental point of view, and trying the effect of loading a large block of indiarubber. He had not had time to refer to the author's paper, in which the reasons were

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* Proc. London Math. Soc. vol. xvii. (1886). See also Love's 'Treatise,' vol. i. pp. 328–330.

given for taking the earth as incompressible. He (Prof. Perry), however, thought that this assumption led to results in contradiction to actually observed facts. Prof. Milne had obtained results which, for want of any other explanation, he had been compelled to attribute to meteorological causes. The reason Dr. Chree had obtained so small a value for the effect of loading by surface-water might be because he had assumed erroneous values for the elastic constants. If he took a value for Poisson's ratio such as we meet with in practice, the effects would be much larger. Prof. Darwin had also investigated the folding of the surface of the earth due to loading. The results obtained by the author with reference to the velocity of waves did not seem quite satisfactory. The small waves which were found, both at Berlin and the Isle of Wight, to precede the main waves coming from an earthquake in Japan, were not accounted for. The wave-velocity in an infinite mass of steel (a very elastic material) was about 6 kilometres per second, which was very different from 12·5 kilometres per second. The author had assumed such values for the elasticity as would give the correct velocity.

The Author, in reply, said that in applying the equations of elasticity to the earth's interior, unless the material were supposed nearly incompressible, one obtained values of the strains too large to be consistent with the fundamental mathematical hypothesis that the squares of strains are negligible. In the case of surface-loading no such restriction was necessary, so far as the surface-layers at least are concerned. The differences between the several numerical estimates for the ratio of gravitational and pressure effects of a surface-load were principally due to the differences in the hypothetical values ascribed to the rigidity. It was his wish to make it clear that the pressure and gravitational agencies treated in detail in the paper were not the only ones likely to affect the level; he had specially called attention to solar heating and possible direct influence of moisture on the foundations of buildings, &c. The reason why for the one wave-velocity so much higher a value was obtained than that Prof. Perry calculated for steel was solely the high value, 24 : 1, found for the ratio of Thomson and Tait's elastic constants m and n.

He knew Prof. Darwin had treated of the phenomena met with in loose earth in some cases, but could not say whether this was what Prof. Perry referred to. He had himself once thought of attempting an application of what Prof. Karl Pearson termed the "equations of pulverulence," as treated in detail by Prof. Boussinesq, but had not done so, partly from a feeling of uncertainty as to their physical value. Supposing these equations satisfactory, they ought to give better results than the equations of elasticity when surface-load was applied to a deep alluvial soil.

V. *On the Effect of Capacity on Stationary Electrical Waves in Wires.* *By* W. B. Morton, *M.A.*[*]

While working recently at stationary electrical waves in wires produced in Blondlot's manner, I was led to make some measurements on the effect produced when a capacity is inserted at a point of the secondary circuit. The positions of the successive nodes were explored in the usual way by a bridge, the indicator being a vacuum tube which was placed across the wires and which showed a maximum of brightness when the bridge was at a node. When two opposite points of the parallel secondary wires were joined to the plates of a small air condenser, the effect was to bring closer together the nodes on the two sides of the condenser, the amount of this shortening of the apparent half wave-length depending on the position of the inserted capacity. The effect was nil when the condenser was at a node, and maximum when it was midway between two nodes. This influence of the increased capacity of the wires is of course of the same nature as the shortening of the wave-length when the wires pass from air into a dielectric liquid. Drude and others have made use of this way of measuring directly the index of refraction of different liquids for the electric waves; but the influence of an isolated capacity does not seem to have been much studied. Salvioni has published[†] some measurements on the effect of capacity inserted at a point between the end condenser and a bridge. When the second condenser was put in it was necessary, in order to restore resonance, to alter the distance of the plates of the end condenser. Von Geitler* got rid of the reflected waves by using a terminal resistance so that no standing oscillations were formed in the wires. He found that when a condenser was inserted it caused reflexion of the waves, and a series of nodes and loops could be found in front of the condenser. Mazzotto † in a recent research has availed himself of the effect of an isolated capacity to produce a gradual change in the wave-length of the oscillations. He uses pieces of wire hung on to the parallel wires.

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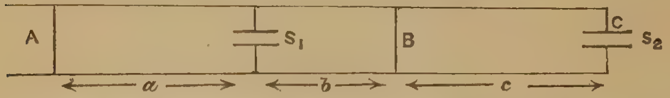
[*] Read April 9, 1897.

[†] *Rend. Acc. Linc.* 1892, pp. 250–253; Wied. *Beibl.* xvii. p. 485.

The theory of electric waves in wires has recently been treated in an exhaustive manner by Drude ‡. His method consists in following out in detail the various reflexions undergone at the bridges by a wave-train which starts from the end of the wires. The state of affairs at a point of the circuit is obtained by summation of a series of separate disturbances due to the different direct and reflected trains. The calculations are rather complicated. In obtaining a formula with which to compare my observations I have used a method adapted from some work of Mr. Heaviside's §. Apart from the actual results obtained, the investigation is perhaps of some interest as showing how easily some problems connected with oscillations in wires can be attacked by this method.

The experiments were made, for the most part, at the end of the parallel secondary wires remote from the oscillator, the arrangement being as shown in the diagram.



C is the end condenser of capacity S_2. S_1 an interposed condenser. The vacuum tube was placed across the plates of

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* Wied. *Ann.* xlix. pp. 184–195 (1893); *cf.* Barton and Bryan, Proc. Phys. Soc. xv. p. 23.

† *Nuov. Cim.* [4] ii. pp. 296–311 (1895); Wied. *Beibl.* xx. p. 392.

‡ *Abhandl. der Sächs. Ges. der Wiss.* xxiii. pp. 64–168 (1896); Wied. *Ann.* lx. pp. 1–46 (1897).

§ 'Electrical Papers,' ii. p. 194 *et seq.*

S_2. The plates of S_1 and S_2 were kept at a constant distance apart while the position of S_1 was varied. A nodal position B of the bridge having been found so that the tube shone, B was left on the wires, and A laid on and adjusted so that the tube remained bright. Then A, B are nodes of the same system. Owing to the finite length of the bridge the potential-difference at its ends is not quite zero, the true node falling at the centre of the bridge. Accordingly, if the bridge B is taken off the wires the nodes will fall a short distance to the left of B—roughly half the length of the bridge. This distance Drude has called the "bridge-shortening." In order to restore maximum brightness in the tube, the bridge A must now be displaced in the same direction through *twice* the "bridge-shortening." The difference in the positions of A, according as B is on or off the wires, gives us therefore a means of finding the correction to be applied to the observed bridge-positions to get the true nodes in the wires.

We want to express that the circuits AB and BC are in resonance. In order to find approximately the period of oscillation of such circuits we can proceed as follows:—Suppose a simple harmonic potential-difference $V_0 \sin nt$ to be kept up between the wires at one end of the circuit, and find an expression for the oscillations produced in the circuit. The amplitude of these oscillations will become infinite when n corresponds to the natural period of the system. A formula has been obtained by Cohn and Heerwagen for a circuit like BC. I have not found any discussion of the circuit AB.

We neglect the resistance of the wires and put S for their capacity and L for their induction per unit-length, both supposed constant. This will be only approximately true as we approach the ends of the circuits. The equations connecting the current C and potential-difference V are

$$-\frac{dV}{dx} = L\frac{dC}{dt}$$

$$-\frac{dC}{dx} = S\frac{dV}{dt},$$

dx being the element of length,

$$\therefore \frac{d^2V}{dx^2} = LS\frac{d^2V}{dt^2};$$

or if V varies as $\sin nt$,

$$\frac{d^2V}{dx^2} = -LSn^2V = -q^2V,$$

where $$q = n\sqrt{LS} = \frac{n}{v} = \frac{2\pi}{\lambda};$$

v is the velocity of radiation, and λ the wave-length along free wires;

$$\therefore\ V = (A\cos qx + B\sin qx)\sin nt.$$

This gives $$-C = \frac{q}{Ln}(A\sin qx - B\cos qx)\cos nt.$$

Take first the circuit BC. Here we have

$$V = V_0\sin nt \text{ when } x = 0,$$

and $$C = S_2\frac{dV}{dt} \text{ when } x = c,$$

$$\therefore\ A = V_0,$$

and $$-\frac{q}{Ln}(A\sin qc - B\cos qc) = S_2n(A\cos qc + B\sin qc).$$

Putting $$\tan\alpha = \frac{LS_2n^2}{q} = LS_2v^2q = \frac{S_2}{S}q$$

we get $$B = A\tan(qc + \alpha),$$

and $$V = V_0\frac{\cos\{q(c-x)+\alpha\}}{\cos(qc+\alpha)}\sin nt.$$

The amplitude becomes infinite when $qc + \alpha = \frac{\pi}{2}$,

$$\cot qc = \tan\alpha;$$

$$i.e.\quad \cot\frac{2\pi c}{\lambda} = \frac{2\pi}{\lambda}\cdot\frac{S_2}{S}. \quad \dots\dots \quad (1)$$

This is Cohn and Heerwagen's formula.

Taking now circuit AB, suppose the impressed potential-difference to act at A.

Let $V_1 = (A_1\cos qx + B_1\sin qx)\sin nt$ between $x = 0$ and $x = a$,
$V_2 = (A_2\cos qx + B_2\sin qx)\sin nt$ between $x = a$ and $x = b$.

The conditions to be satisfied are :—

$$\text{at } x=0, \quad V_1=V_0 \sin nt,$$

$$\text{at } x=a, \quad V_1=V_2,$$

and

$$C_1-C_2=S_1\frac{dV}{dt}$$

$$\text{at } x=a+b, \quad V=0.$$

Putting in the values we get

$$A_1=V_0,$$

$$A_1 \cos qa + B_1 \sin qa = A_2 \cos qa + B_2 \sin qa$$

$$=\frac{1}{t}\{(-A_1+A_2)\sin qa+(B_1-B_2)\cos qa\},$$

$$A_2 \cos q(a+b)+B_2 \sin q(a+b)=0,$$

t has been put for $\frac{qS_1}{S}$.

On solving these equations for the constants we find

$$V_1=\frac{\sin q(a+b-x)-t\sin q(a-x)\sin qb}{\sin q(a+b)-t\sin qa\sin qb}V_0\sin nt,$$

$$V_2=\frac{\sin q(a+b-x)}{\sin q(a+b)-t\sin qa\sin qb}V_0\sin nt.$$

Equating the denominator to zero we find

$$\cot qa+\cot qb=t,$$

or

$$\cot\frac{2\pi a}{\lambda}+\cot\frac{2\pi b}{\lambda}=\frac{2\pi}{\lambda}\cdot\frac{S_1}{S}. \qquad (2)$$

This formula, which connects the frequency of the oscillation with the position and capacity of the condenser, might have been deduced from formula (1). For we can imagine the capacity S_1 divided into two parts, σ and σ', in such a way that when these parts are attached to the parts a and b of the circuit respectively, the two partial systems oscillate independently with the same frequency. We have thus from formula (1),

$$\cot\frac{2\pi a}{\lambda}=\frac{2\pi}{\lambda}\frac{\sigma}{S},$$

$$\cot\frac{2\pi b}{\lambda}=\frac{2\pi}{\lambda}\frac{\sigma'}{S}.$$

On adding we get equation (2).

In seeking to test the agreement between the theory and the observations, formula (2) was written

$$\frac{\lambda}{2}\left\{\cot\frac{2\pi a}{\lambda}+\cot\frac{2\pi b}{\lambda}\right\}=\frac{\pi S_1}{S}=\text{constant.}$$

A small error in observation of the node position causes an error in the left-hand side very large in proportion, and of amount varying with the position of the condenser in the circuit. In view of this the method adopted was to find the mean value of $\frac{S_1}{S}$ from the observations and, using it, to calculate the values of b corresponding to each a. A comparison of the observed and calculated values of b shows a sufficiently good agreement, the discrepancy being greatest when the condenser is too near a bridge.

One set of observations involving only the circuit AB and formula (2) were taken on a small apparatus at the end near the oscillator. One bridge was kept fixed and the wave-length was the same throughout. Thus only the quantity b was liable to the error in determining the node. The uncertainty in this determination amounted to about 5 millim. The wires were about 15 millim. apart, and the condenser consisted of two small copper strips, 1 centim. by 5 centim., hung on the wires. The half wave-length was 37·5 centim. The following table gives the results measured in centimetres:—

TABLE I.

a.	b. Calc.	b. Obs.
2·0	34·7	33·1
6·0	19·8	17·5
8·0	13·4	12·9
10·0	10·3	10·5
12·0	8·5	8·7
14·0	7·7	7·9
20·0	5·9	5·9
30·0	3·7	3·9
33·0	2·7	3·1

The other observations were made at the end of the wires of a large oscillator in the manner already described. In the calculation both formulæ had to be used. First, a measurement without the interposed condenser gave directly a value of λ for a given value of c. Then from formula (1) by use of these values the constant $\frac{S_2}{S}$ was found. In working up the subsequent observations with condenser S_1 in position, knowing c gave λ from formula (1), then λ, a, b in (2) gave a value of $\frac{\pi S_1}{S}$. Using the mean of such values the calculated b's were got as already indicated.

The parallel wires were about 20 m. long, 2 millim. diameter, and 10 centim. apart. The plates of condensers S_1 and S_2 were of 8½ and 20 centim. diameter respectively. Uncertainty of node position from one to two centimetres.

TABLE II.

Distance of plates of S_1, 0·5 centim.; of S_2, 3·7 centim.

Without S_1, $c=44{\cdot}5$; $a+b=\frac{\lambda}{2}=302{\cdot}5$. Hence $\frac{S_2}{S}=13{\cdot}2$.

c. Obs.	$\frac{\lambda}{2}$. Calc.	a. Obs.	b. Calc.	b. Obs.
45·1	304·6	4·4	299·5	296·0
51·3	326·6	54·9	82·5	77·8
50·7	324·5	79·7	55·3	54·7
50·1	322·4	113·5	43·3	42·9
48·5	316·8	161·5	34·4	35·7
46·3	308·9	224·3	28·1	27·9
44·7	303·2	290·3	9·3	10·1

TABLE III.

The same capacities, a different node-system.

Without S_1, $c=54{\cdot}5$; $\frac{\lambda}{2}=337{\cdot}5$ giving the same value, $\frac{S_2}{S}=193{\cdot}2$.

c. Obs.	$\frac{\lambda}{2}$. Calc.	a. Obs.	b. Calc.	b. Obs.
61·5	360·6	54·9	179·5	169·9
62·9	365·0	79·7	55·3	54·7
62·7	364·4	100·9	65·7	67·3
60·9	358·7	149·1	49·0	50·1
56·3	343·3	278·7	26·4	27·2
55·3	340·0	305·7	19·0	19·9

TABLE IV.

Distance of plates of $S_1 = 1$ cm., of S_2 unchanged $= 3·7$ cm.
Same node-system as in Table III.

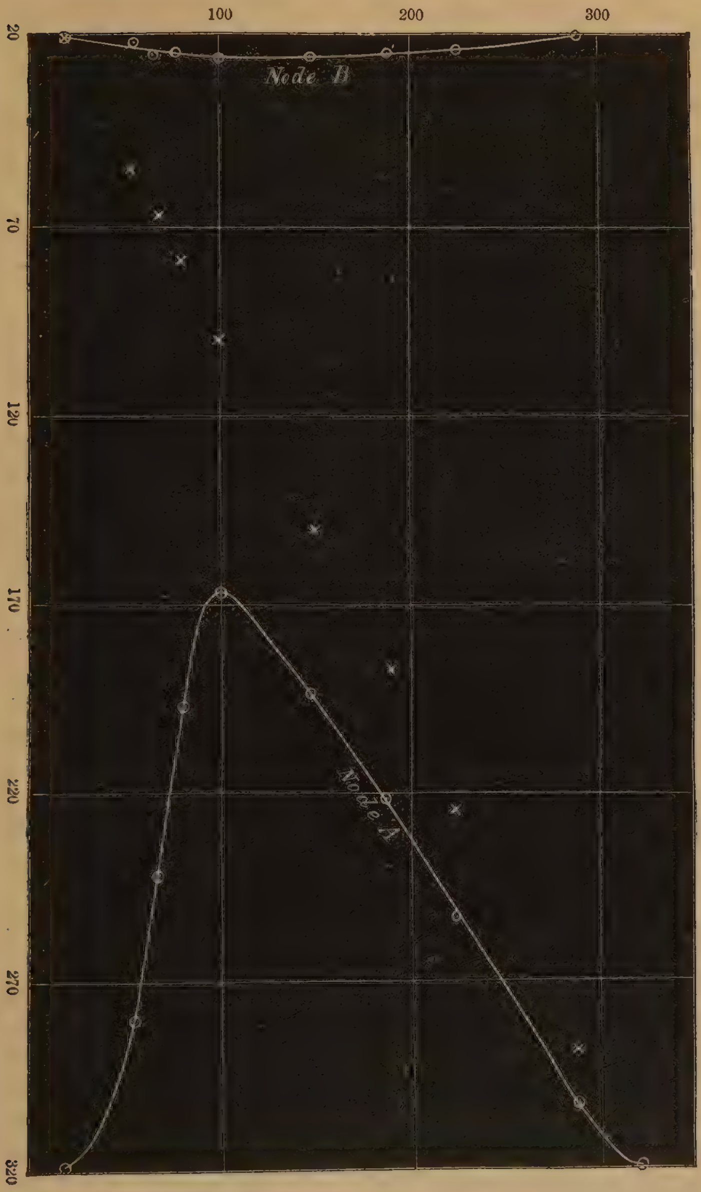
c. Obs.	$\frac{\lambda}{2}$. Calc.	*a.* Obs.	*b.* Calc.	*b.* Obs.
58·1	349·4	83·9	192·8	187·6
58·5	350·6	148·1	110·7	111·5
57·5	347·1	194·5	81·3	81·0
56·5	344·0	240·5	60·0	61·0
55·3	340·0	288·2	36·4	36·8
55·0	339·0	321·3	15·4	15·2

TABLE V.

Distance of plates of $S_1 = ·5$ cm., of $S_2 = 6·2$ cm.
Without S_1, $c = 60·0$; $\frac{\lambda}{2} = 300·0$; $\frac{S_2}{S} = 131·4$.

c. Obs.	$\frac{\lambda}{2}$. Calc.	*a.* Obs.	*b.* Calc.	*b.* Obs.
62·8	307·8	27·6	233·1	231·7
65·9	316·4	36·6	186·5	180·0
65·5	315·3	49·1	113·2	122·9
67·0	319·4	68·0	77·2	73·5
66·0	316·7	118·8	48·3	49·7
65·5	315·3	157·1	40·4	40·5
63·9	310·9	196·6	33·7	34·6
60·5	301·4	262·9	19·6	20·8

The general course of the changes referred to is best seen from the annexed figure, which corresponds to the last series of observations. The ordinates are distances of the inserted condenser from an arbitrary origin near the end of the wires. the abscissas of the points on the two curves are the distances of nodes B and A from the same origin on twice the scale, the crosses between the curves showing the position of the condenser. It will be seen that the apparent half wave-length, or distance A B, is least when the condenser bisects the distance. When the condenser coincides with either node, A B is the full half-wave. When the condenser passes outside A B, the curve of A turns in again.


100
200
300
20
70
120
170
220
270
320
Node B
Node A

I have to express my thanks to Prof. Ebert of Kiel, in whose laboratory and under whose kind direction the experiments were carried out of which these observations form part.

Queen's College, Belfast,
27th February, 1897.

VI. *Liquid Coherers and Mobile Conductors.*

By Rollo Appleyard[*].

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In a communication[†] made three years ago to the Physical Society, I described some experiments illustrating the change in electrical resistance of certain complex bodies under the influence of oscillatory discharges. All the substances dealt with were solids; and the coherence was invisible. The change of condition had therefore to be demonstrated either by measuring the resistance before and after discharge, or by connecting the coherers permanently in series with a battery and galvanometer.

The three experiments now brought before you have regard to "coherers" formed of liquid dielectrics, and mobile conductors. By choosing a transparent dielectric and an opaque conducting substance, it is possible to examine the process of coherence by direct observation. But it may be well to premise that the similarity of *results* obtained with solid and liquid "coherers," respectively, in no way proves a similarity of *process*. The two sets of phenomena are probably related, but are not necessarily identical. The term "dielectric" is here to be understood as signifying merely a substance of low conductivity.

Experiment 1.—A glass tube about eighteen inches long, and half-an-inch wide, is sealed at one end and corked at the other. Platinum electrodes are inserted at each end. The tube is nearly filled with about equal volumes of paraffin-oil and mercury. If it is laid upon a flat table and shaken, horizontally, for a few minutes, the mercury breaks up into small spheroids; and, by a little manipulation, these can be disposed as a chain of particles lying evenly between the platinum electrodes. The resistance of the chain of mercury spheroids, measured under these conditions, is several megohms.

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[*] Read March 26, 1897.

[†] "Dielectrics," Phil. Mag. xxxviii. p. 396 (1894); Proc. Physical Society, xiii. p. 155 (1895).

If we now connect the electrodes to a battery of about two hundred volts, the whole regime is suddenly altered. At the moment of applying the current, the spheroids of mercury, within the tube of oil, are visibly impelled, as though a mechanical tap had been administered to the glass; and, almost simultaneously, they coalesce into large globules. The resistance is now represented by a few ohms.

Exactly the same result can be brought about by supporting the tube near a Hertz oscillator; or, still more simply, by passing a spark into one or other, or both, of the electrodes. In order to retard the spontaneous coherence of the mercury, resulting from mutual pressure of the spheroids, it is well to keep the tube horizontal. If, however, it is desired quickly to convert a body of mercury from the subdivided to the ordinary state, sparks may be passed into the tube while it is more or less vertical. The running-together of subdivided mercury is more leisurely to be observed with large globules. These form separate, elongated, conductors. The way in which they unite will be referred to in describing Experiment 3.

Experiment 2.—A glass tube, similar to the first, but somewhat wider, is nearly filled with a mixture of paraffin-oil and water, and vigorously shaken. I propose to call this a "rain" tube. If it is kept at rest, the oil, in the common course of events, floats to the top in a few minutes. The "rain" tube, however, shows that the separation, especially towards the final stage, is accelerated by the passage of a spark, or by a direct current from a battery of about a thousand volts. If the conditions are right, the water particles suspended in the oil cohere, at the moment of electrification, to form larger drops. The frictional resistance to falling is thereby diminished, and the water is consequently precipitated in and through the oil. It may sometimes be seen descending in a rapid succession of globules, precisely as large rain-drops are precipitated after thunder. About equal parts of oil and water is a good proportion. The containing vessel may be either a tube or a flask; it should not be more than three-quarters full of liquid. This free space facilitates the mixing when the tube or flask is shaken. The phenomenon is rendered much more striking by colouring the oil with alkanet-root. I have to thank Prof. McLeod for suggesting this pigment.

Experiment 3.—The behaviour of a mobile conductor, when electrified in a partially conducting liquid, is readily examined by pouring a little mercury into a flat photographic dish containing a stratum of paraffin-oil and water. The presence of the oil is necessary to prevent the mercury from running together too freely of its own accord. A battery of from, say, one volt to two hundred is required, and a pair of wires to dip into the dish. A reversing-key, such as is used for cable transmission, may be included in the circuit.

Suppose we begin with a large globule of mercury in each of any pair of corners of the photographic dish, several inches apart; and let the globules be connected one to each pole of the battery, by means of the dipping wires. A momentary tap of the key causes instantaneous deformation of the mercury in each corner, especially of that connected to the negative pole; and there is evident attraction between the globules. Sometimes the mercury gets into a lethargic condition; but it can always be roused by mechanical agitation of the surface. Now let the current be kept on for a few seconds; the negative globule sends forth a tentacle towards the positive globule, the length of the tentacle depending upon the current and the distance between the globules. Under favourable circumstances it may extend from corner to corner, and thus establish contact; or *fissure* may occur, the tentacle breaking into spheroids; and these spheroids may cross over between the globules. This is the order of things usually to be observed, but the action is sometimes erratic.

Let us now bring back the scattered globules of mercury to their respective corners, and distribute a few isolated spheroids in the interspace. In addition to the effects previously noticed at the terminal globules, we now see that, when the current is applied, each intermediate spheroid extends a " finger " towards the positive globule. This is the process of mobile coherence; the short " fingers," or long "tentacles," form links between consecutive spheroids, and finally a complete conducting circuit is established.

By successive applications of the current, any elongated bodies of mercury between the terminal globules can be made to creep along like caterpillars; the successive forward motions of the tentacle, or tail, cause a corresponding retrogression of the globule as a whole.

Any small spheroids scattered about the dish may be urged in a direction depending upon the direction of the successive current impulses; and a "finger" will always appear on the side towards the positive electrode. So that by choosing a convenient stray spheroid, and operating a battery-reverser as a transmitting-key, a telegraphic receiver is improvised from no other apparatus than a drop of mercury and a little oil. By some such means the awakening genius of primitive man may have contrived all the subtle machinery of a telegraph-instrument upon the smooth surface of an oyster-shell.

DISCUSSION.

Prof. RAMSAY said he had once attempted to facilitate churning by the application of 8 or 9 volts to some milk. He thought the cream came a little faster, but it turned sour very quickly.

Prof. FITZGERALD thought that the effects observed in experiment 3 were the result of current, and not of electrostatic changes; and he would like to know the value of the actual current used. There was no doubt that the motions were due to variations in capillarity.

Mr. SHELFORD BIDWELL asked how the mercury was formed into spheroids in the tube in experiment 1.

Mr. APPLEYARD, in replying to Prof. Fitzgerald, said it was not easy to define the circuit, as the terminal-globules were rather capricious, but he would try and measure the current in some particular case. The mercury-tube in experiment 1 was shaken in a horizontal plane; the operation took about ten minutes. Equal volumes of mercury and oil was a good proportion. One quarter of the length of the tube should be left as an air-space.

VII. *The Thermo-electric Properties of some Liquid Metals.*

By William Beckit Burnie, *lately* 1851 *Exhibition Science Research Scholar, Nottingham* *.

The object of the experiments here to be described was to compare the thermo-electric properties of solid metals with those of the same metals when melted. The four metals employed, tin, lead, bismuth, and mercury, were each thermo-electrically compared with copper, the tested metal being contained in a hard glass tube, so that the observations could be pushed to temperatures considerably above those of the melting metals, and the changes in the thermo-electric properties during the process of melting observed. Two sets of experiments were made, the first set with the greater part of the metal under test at ordinary temperatures, and the second set with all the metal under test at high temperatures.

In the first set of experiments with tin, lead, and bismuth, the glass tube containing the metal was W-shaped, the metal filling the central part, but only rising about one-third of the height in the outside limbs. To fill this tube one end was dipped into a crucible full of the melted metal, which was then allowed to cool. The crucible and tube were immersed in a bath of linseed oil, which was raised to a temperature above that required to melt the metal. When the metal was quite liquid, air was withdrawn from the upper end of the tube till the metal had risen to the right height. The tube was next slowly withdrawn from the still hot oil, to prevent cracking due to the freezing of the metal, and allowed to cool. The superfluous metal was then melted out of the open limbs. The two copper-metal junctions were to be in the two open limbs of the tube, and a number of thick copper wires were so arranged round the outside of the tube as to conduct away the heat from all parts not in the immediate neighbourhood of the hot junction. By this means all the metal except a small portion was kept solid, and the level of the melted metal was always higher in the open than in the closed limb. This was necessary, as the solid metal was not a satisfactory cork for the tube, and a slight liquid pressure was needed to keep good contact between the liquid and solid metal.

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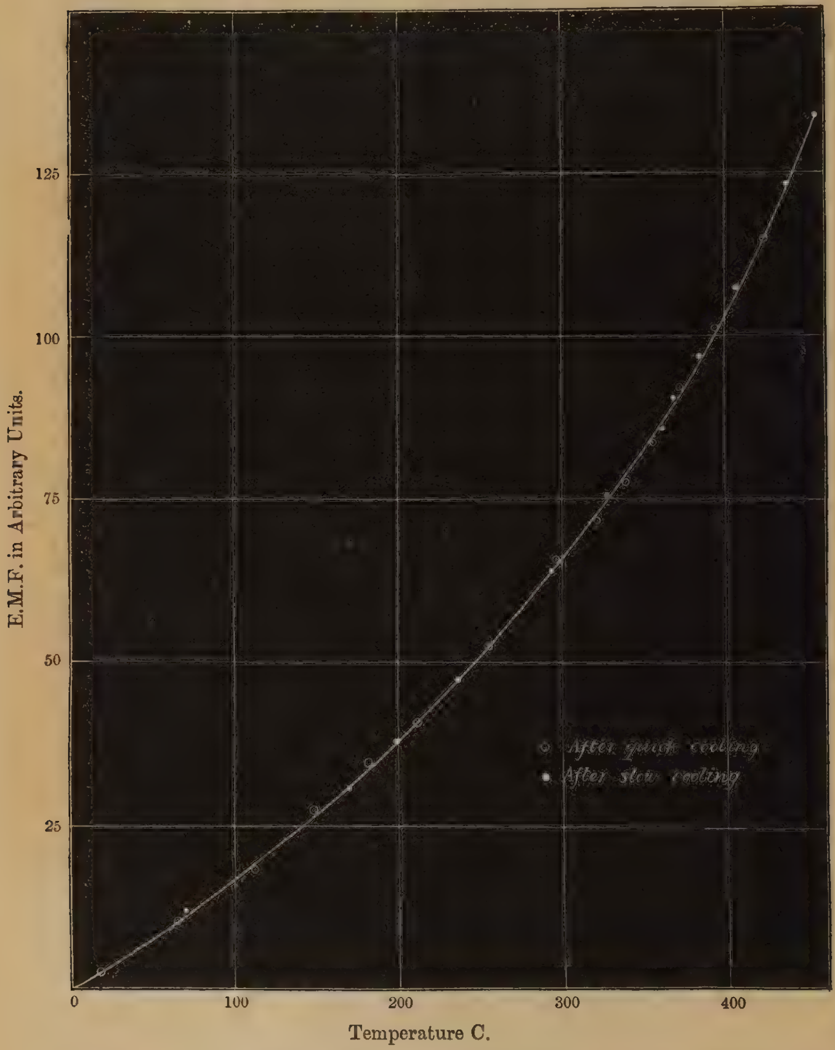
* Read February 26, 1897.

The temperature-differences between the hot and cold junctions were measured with a thermo-element of platinum and a platinum-rhodium alloy (Pt 90 per cent., Rd 10 per cent.). This thermo-element was calibrated between the temperatures of 0° C. and 263°·5 C. by comparison, in linseed oil, with a Reichsanstalt standardized thermometer. The calibration curve was then extrapolated as far as 430° C. by means of an equation. For reasons which will be explained later, it was useless to adopt any more accurate method than this for the first set of experiments. The thermo-element was insulated within and from without with asbestos paper, and each junction was bound to one of the standard copper wires. These copper wires were also insulated with asbestos, so that only their ends, which were in close proximity to the junction of the temperature-measuring thermo-element, were uncovered.

Some experiments were made to determine the electrical insulating properties of asbestos paper at high temperatures, and it was found that the errors introduced by the electrical conductivity of the supposed insulator were, up to 600° C., negligibly small. Other experiments were made to determine whether or not the temperature of the junction in asbestos differed greatly from the temperature of the metal surrounding the asbestos. One junction of the temperature-determining thermo-element was immersed in melted lead, and the other in melting ice. The parts of the wires just outside the lead were alternately heated with a burner, whereby the temperature-difference between the junction and the metal would be decreased, and cooled with an air blast, whereby the temperature-difference would be increased. Readings of the temperature-difference between the two junctions were meanwhile taken; and it was found that by varying the temperature of the leading-in wires the temperature of the hot junction was not altered by more than $\frac{1}{10}$ of a degree. As the leading-in wires were hotter, when warmed by the flame, than the melted lead, $\frac{1}{10}$ degree was greater than the greatest error introduced during the experiments by the heat-insulating capability of the asbestos paper.

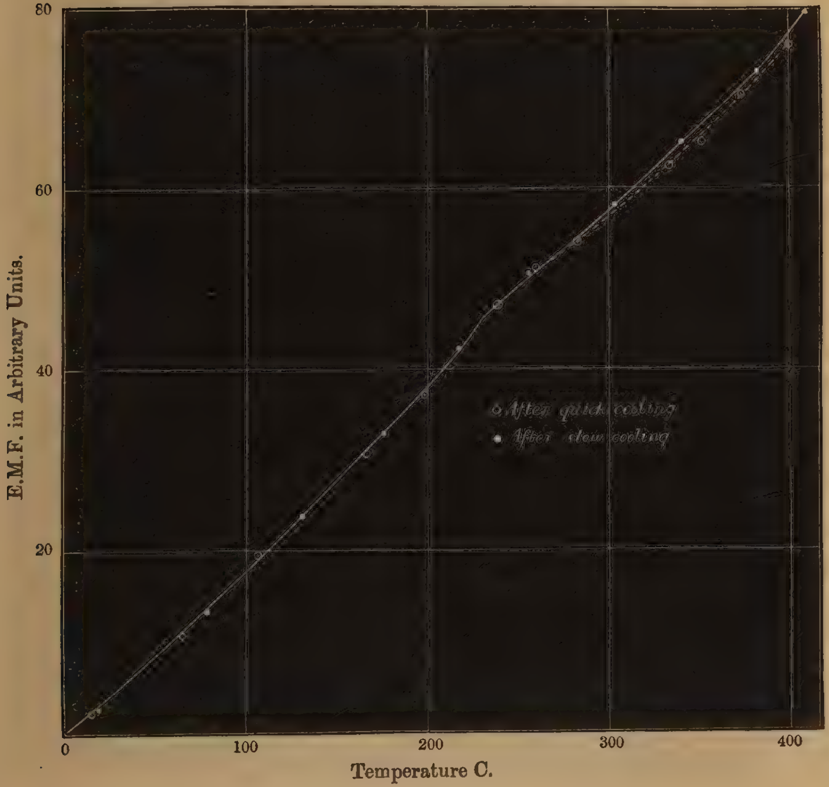
Fig. 1.—Copper-Lead Thermo-element.



Each of the copper wires with its thermo-element junction was melted into the metal in its respective limb, and held in the centre of the tube by asbestos paper. One limb of the W was heated in a sand bath, and the other was kept cold in melting ice.

The two standard copper wires and the two free wires from the temperature-determining thermo-element were led to a paraffin switch-block, so that either pair could be connected, through a reversing key and a resistance, to a reflecting galvanometer. This resistance was so great as to render no correction necessary for the resistance variations, due to heat, in the circuit. For each reading the temperature of the hot junction was made nearly constant; then, using each thermo-electric combination alternately, galvanometer deflexions were noted with the current direct, and the current reversed. After the necessary correction, the mean of the readings of

Fig. 2.—Copper-Tin Thermo-element.



the platinum-platinum-rhodium thermo-element gave the temperature, and the mean of the readings of the thermo-element under test gave the E.M.F. in arbitrary units.

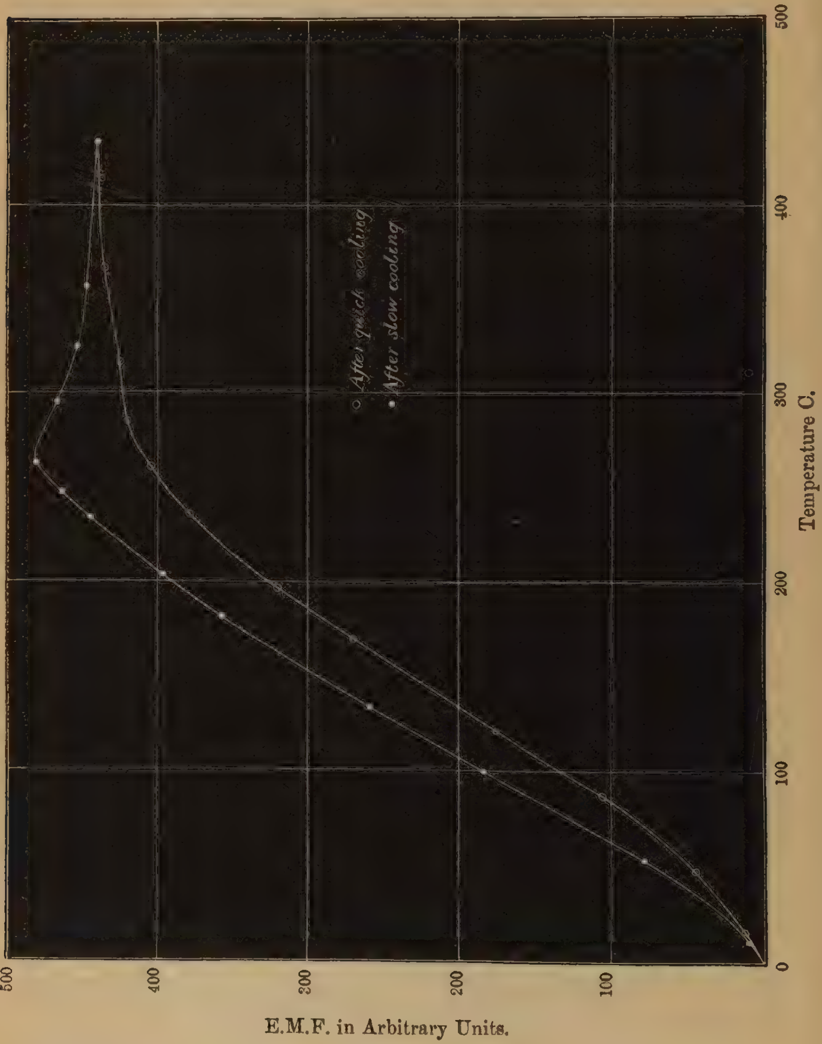
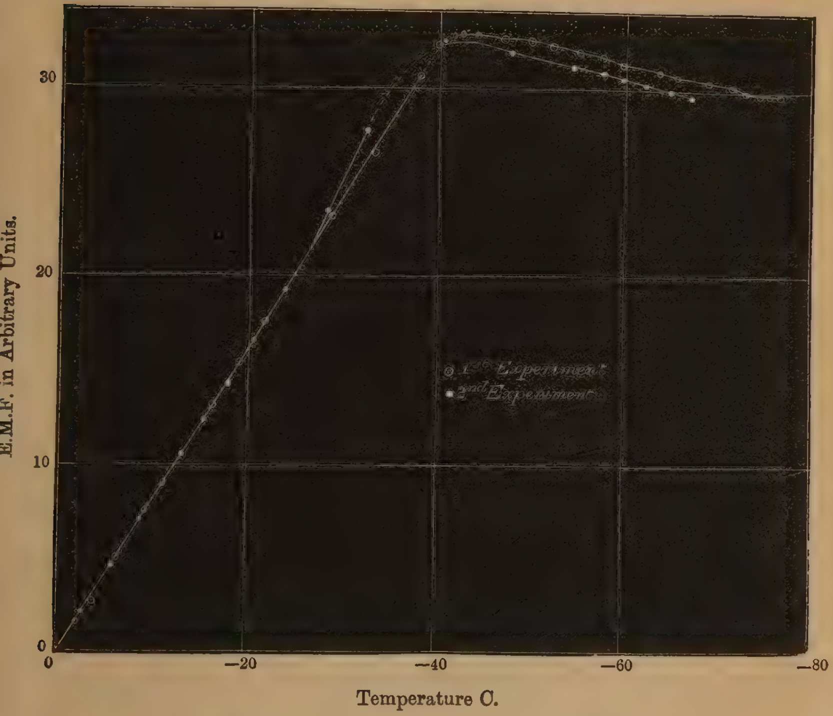


Fig. 3.—Copper-Bismuth Thermo-element.

After each complete experiment, the constant of the temperature-measuring thermo-element was determined by removing the crucible containing sand, putting in its place one containing melted lead, and allowing the lead to cool, meanwhile taking readings.

The curves for copper-lead, copper-tin, and copper-bismuth, which were obtained by this method, are given in figs. 1, 2, and 3 respectively. The abscissæ are temperatures and the ordinates E.M.F.'s in arbitrary units. These curves will be discussed when the mercury experiments have been described.

Fig. 4.—Copper-Mercury Thermo-element.



The W-shaped tube was not suitable for the application of a freezing-mixture, and so in the mercury experiments the junctions were each in a small test-tube about half filled with the metal. An inverted U-tube also full of mercury, with one end dipping into the metal in each of the test-tubes, connected the two. One of the test-tubes was surrounded by melting ice, and the other was cooled with a mixture of solid carbon dioxide and ether. A mass of copper wire was wrapped round the tube to be cooled, in order that it might warm up slowly after the evaporation of the carbon dioxide.

As the temperatures to be measured approached the neutral point of the thermo-element used in the last experiments, an iron-constantan thermo-element was employed to measure the temperature-differences between the two junctions. This thermo-element was calibrated by the observation of the galvanometer deflexions when one junction was in melting ice and the other junction first in melting mercury and then in boiling carbon dioxide and ether. As these two deflexions were nearly proportional to their respective temperature-differences, the calibration curve equation was assumed to be of the form, deflexion $=at+bt^2$, where t is the temperature, and a and b constants. These two constants were determined from the two observations, and the calibration curve for the thermo-element was plotted from the equation. The element and the standard copper wires were insulated as in the last experiment, and bound to the glass tubes dipping into the mercury. The freezing-mixture was contained in a double-walled test-tube, the inner and outer walls being separated by cork distance-pieces, and the whole being wrapped in flannel. The electrical connexions were the same as in the last experiment.

In the experiments, the tube containing the cold junction was cooled as far as possible with the freezing-mixture, and allowed to warm slowly up. Readings of the galvanometer were meanwhile taken every thirty seconds, as follows:—First, a reading from the temperature-determining thermo-element with the current direct; then a reading from the same element with the current reversed; then a reading from the copper-mercury element with the current direct; and lastly, a reading from this element with the current reversed. The cycle then began again. From the means of each pair of readings, reduced, two curves were plotted, one giving the relation between temperature and time, and the other giving the relation between E.M.F. in arbitrary units and time.

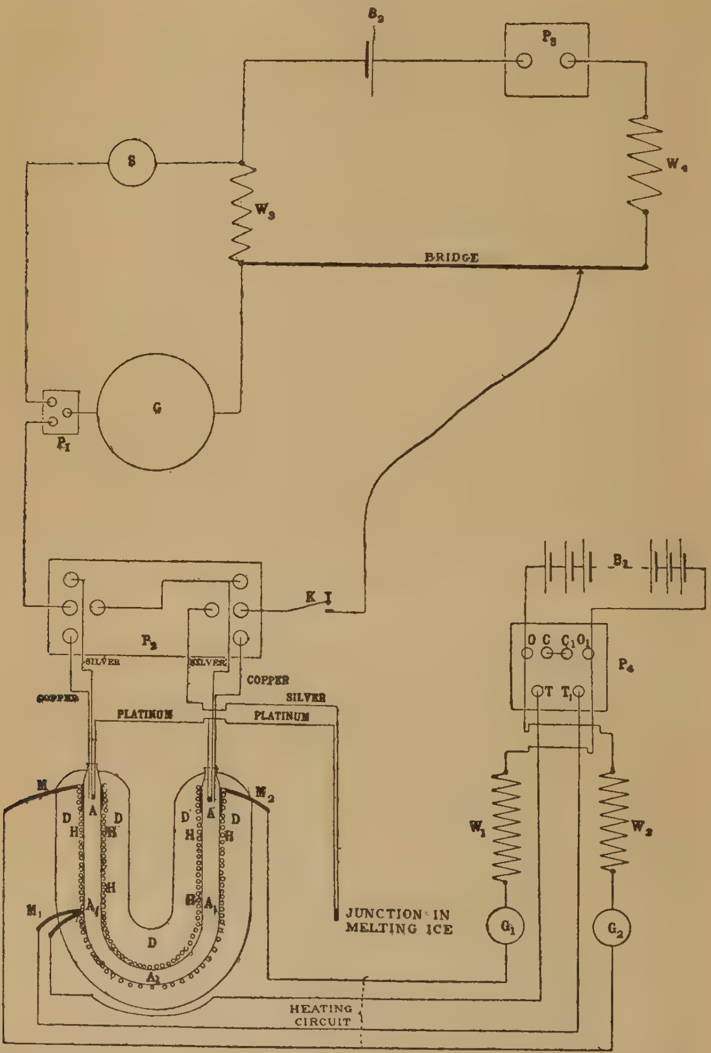
The curves (fig. 4) give the relation between E.M.F. and temperature deduced from these two curves, for two experiments.

In figs. 2, 3, 4 two curves are in each case given. These two curves were obtained from the same metal. The effect is most marked with bismuth (fig. 3). At the conclusion of one experiment the apparatus was cooled very rapidly by removing the sand and blowing air on to the tube containing the melted metal. Another set of readings was then taken, with the temperature of the hot junction rising, and the lower of the two curves in fig. 3 was obtained. At the conclusion of this experiment the apparatus was cooled very slowly, about two hours being occupied, and the result of the next experiment was the upper curve. It was found possible, moreover, with intermediate rates of cooling to produce intermediate curves. This effect is probably due to the variations in the crystalline structure of the metal under test, dependent on the rapidity of its solidification; and it may be that the various thermo-electric properties which have been observed for different pieces of bismuth are largely due to this same cause. With tin the effect was less marked, and with lead it was unnoticeable. In the mercury experiments it was not possible to regulate the rate of cooling; but the two most widely differing curves obtained are given in fig. 4.

As this effect of its previous treatment upon the metal rendered accuracy impossible, the experiments were repeated with three of the metals, viz. tin, lead, and bismuth, with a method so arranged that the whole mass of the metal to be tested was in a molten state, and thus homogeneous.

The apparatus is shown in fig. 5. The metal was contained in a U-tube A A_1 A_1 A, and was heated by a current of electricity in german-silver wires H H H. The metal was introduced into the tube by means of a glass pipette, both tube and pipette being warmed in a bunsen flame. The part of the glass tube where the junctions were to be, A A, was covered with copper to render the heat distribution more uniform. The german-silver wire was insulated with asbestos paper, and wound spirally round the tube in two sections, M M_1, M_1 M_2. Over this were wrapped several layers of asbestos paper and asbestos string, DD. The connexions of the heating circuit are shown in fig. 5. In the paraffin switch-block, P_4, if T C and T_1 C_1 were connected, the same current from the heating battery passed through

Fig. 5.



both ammeters, G_1 G_2, both resistances, W_1 W_2, and both sections of the heating coil. This caused a saving in current when it was merely required to melt the necessary connexions into the metal, or to burn out the asbestos insulation. If, however, T O was connected the current passed through the ammeter G_1, the resistance W_1, and the long section of the heating coil; and if T_1 O_1 was connected the current went through the other ammeter, resistance, and heating-coil section. By adjusting the resistances W_1 and W_2 the temperatures at the junctions could be varied as required. The diameter of the german-silver wire was 0·5 mm., and the largest current required was 4 amperes. The U-tube is shown about one half actual size.

For measuring the temperatures a platinum-silver thermo-element was employed, as this gave E.M.F.'s of the same order of magnitude as those to be measured. This element had three junctions, as shown in fig. 5, so that either the temperature of one of the junctions or the temperature-difference between the two junctions of the thermo-couple under test could be measured. The thermo-element was calibrated with linseed oil up to 288° C.; and one point on the calibration curve, at 441°·4 C., was determined with sulphur vapour. Each limb was calibrated separately, but they were found to be similar. The part of the calibration curve from 288° C. to 430° C. was plotted from the equation :—

$$\text{E.M.F. (in volts)} \times 10^5 = 0{\cdot}1407\, t + 0{\cdot}001227\, t^2.$$

This equation was obtained from the point on the curve determined with sulphur, and the point where $t = 220^\circ{\cdot}7$ C. on the curve as drawn in between 0° C. and 288° C. By the application of the method of least squares to all the calibration observations a somewhat different equation was obtained, viz. :—

$$\text{E.M.F. (in volts)} \times 10^5 = 0{\cdot}1463\, t + 0{\cdot}001238\, t^2\,;$$

but as the calibration curve was not exactly a parabola the upper equation was judged the better.

The junctions and standard copper wires were insulated as in previous experiments and melted into the metals. The two standard wires and the three free wires from the temperature-determining thermo-element were led to a paraffin switch-block, P_2, which was connected to an arrangement of a galvanometer, G, a standard cell, S, a key, K, a bridge battery, B_2, resistances, W_4 and W_3, switch-blocks, P_1 and P_3,

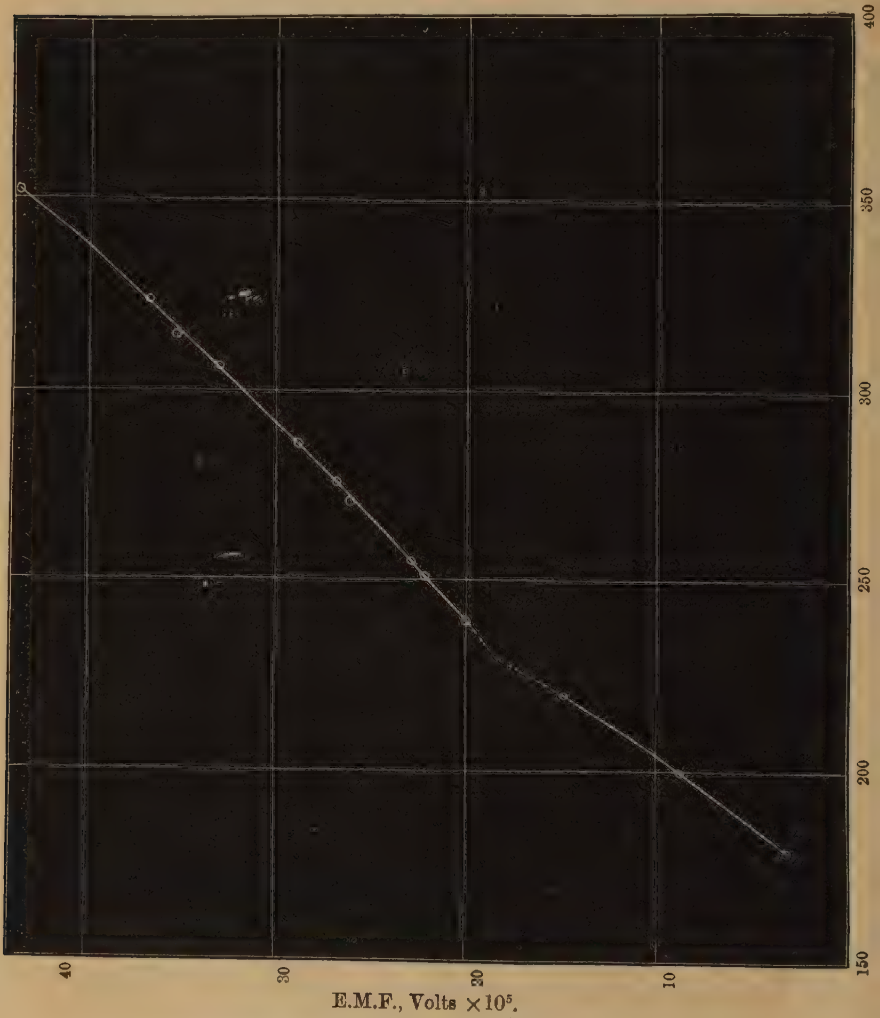
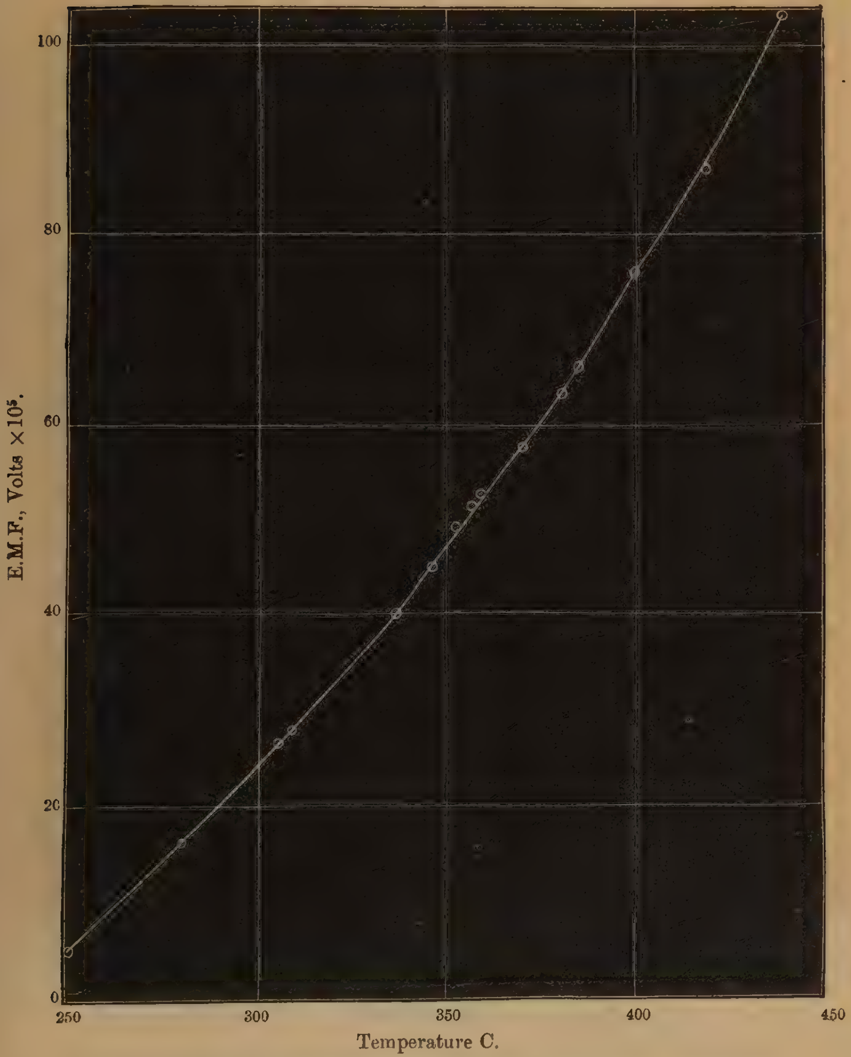


Fig. 6.—Copper-Tin Thermo-element.

and a metre bridge, as shown in fig. 5, for measuring the E.M.F.'s.

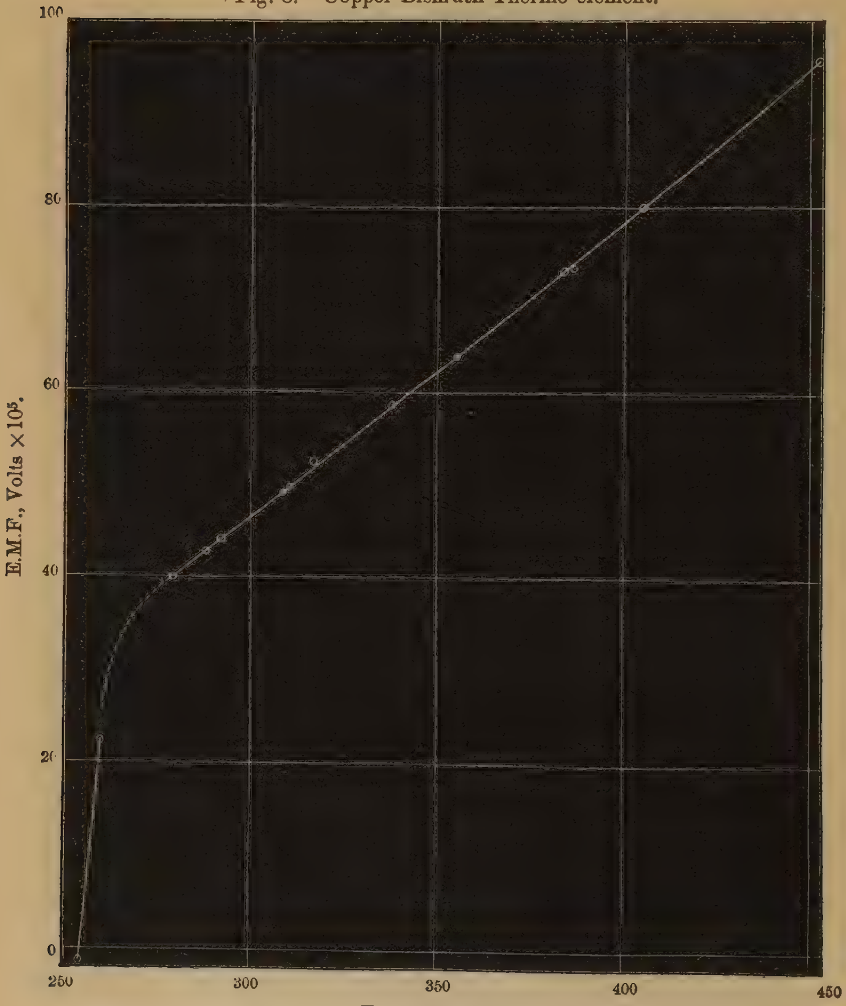
The current in the long heating section was adjusted before each reading till the temperature of the fixed temperature junction was at a certain point, slightly above the melting-point of the tested metal. Readings were taken with the temperature of the variable temperature junction both above and below the melting temperature of the metal. With tin and lead this latter was easy, but with bismuth, owing to

Fig. 7.—Copper-Lead Thermo-element.



the crystallization effect before mentioned, the readings were rather uncertain. Above the melting-point, however, they were perfectly constant.

Fig. 8.—Copper-Bismuth Thermo-element.



The curves plotted from the results of the experiments by this method are given in figs. 6, 7, and 8 respectively.

In each of these curves we see that in a small variation of temperature, about the melting-point, there is a considerable change in the direction of the thermo-electric curve. The effect is smallest with lead, with tin it is larger, and with bismuth it is very remarkable; that metal changing, during melting, from an exceedingly active thermo-electric metal to one very similar to lead in its thermo-electric properties. With mercury also (fig. 4) we see that a great change takes place at the melting-point. This indicates that there is a difference between the Peltier effect for the solid and for the melted metal. But the change in curvature is not sufficiently marked for us to be able to say whether the specific heat of electricity remains proportional to the absolute temperature. To decide this point a direct determination of the Thomson effect is needed.

The worked-out observations from which the curves in figs. 6, 7, and 8 were plotted are given below.

Copper-Tin Element. (Fig. 6.)

Temperature of constant temperature junction 239° C.

Temperature of variable temperature junction.	E.M.F. in volts × 10^5.
255·8	2·86
276	7·01
286·1	9·0
306·1	13·2
323·2	16·92
353·7	23·63
251	2·0
272·2	6·35
315·1	15·45
219·7	−5·1
200·1	−11·15
179·5	−16·86

Copper-Lead Element. (Fig. 7.)

Temperature of constant temperature junction 336°·1 C.

Temperature of variable temperature junction.	E.M.F. in volts × 10^5.
346·1	5·215
357	11·13
370·6	17·92
385·2	26·5
400·5	35·86
418·3	47·4
440·3	63·2
359·1	12·25
352·8	8·5
380·2	23·75
310·3	−11·9
307·3	−13·4
280·5	−23·6
251	−35

Copper-Bismuth Element. (Fig. 8.)

Temperature of constant temperature junction 278°·5 C.

Temperature of variable temperature junction.	E.M.F. in volts × 10^5.
287	2·6
291·4	3·97
308	9·01
315·7	12·05
356·7	24·5
385·1	33·68
405·9	40·15
453·6	56·25
388·3	34·18
339·2	18·94
309·2	9·73
259	−18·01
254·6	−41·2

In each case a constant has been added to the E.M.F.'s before plotting.

These experiments were conducted in the Physical Laboratory of the Eidgenössisches Polytechnikum, Zürich, under the supervision of the director, Prof. Dr. H. F. Weber, to whom my best thanks are due.

VIII. *On the Photography of Ripples.* *By* J. H. Vincent, *B.Sc., A.R.C.Sc., Assistant Demonstrator in Physics at the Royal College of Science, London, S.W.**

[Plates I.–III.]

Many of the phenomena described in this paper have been exhibited at public lectures by Mr. C. V. Boys, using the stroboscopic method, which was first applied to the study of ripples by Lord Rayleigh. I am indebted to Mr. Boys for having recommended to me the work of photographing these effects, and also for many valuable suggestions.

Lord Kelvin defines a ripple as a wave whose length is less than that of the wave which is propagated with the minimum velocity. For ordinary mercury, waves less than 1·3 centim. long are ripples. Capillary ripples are those whose length is so small as to render negligible, in the value of the velocity squared, the term due to gravity. These definitions are rendered clearer by reference to Mr. Boys's Logarithmic Wave Chart. The portion of the curve, representing the relation of the velocity and wave-length, to the left of the point of minimum velocity, refers to ripples. The straight-line portion to the left of the chart represents capillary ripples.

In order to obtain ripples it is necessary to use vibration-frequencies above a certain value. Thus in the case of ordinary clean-looking mercury, with its damp and probably greasy surface, the surface-tension of which may be between 300 and 400 C.G.S. units, a frequency of about 15 per second

* Read February 26, 1897.

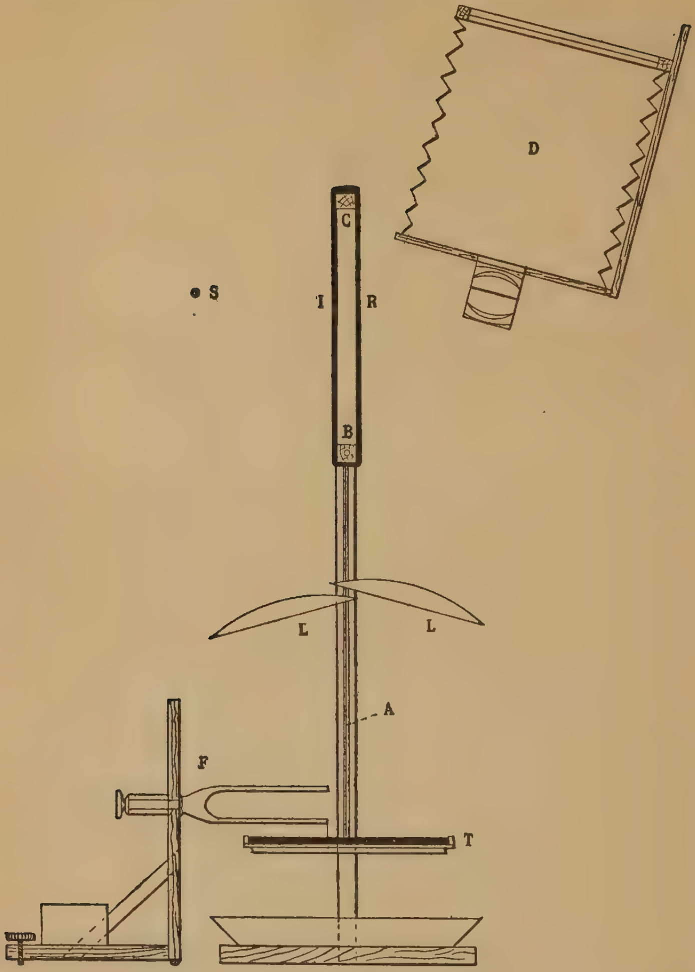


Diagram of Apparatus. $\frac{1}{10}$ nat. size.

T. Mercury trough.
A. Standard of retort-standard.
B. Wooden block through which passes the bent down portion of A.
C. Upper bar of wooden stand.
IR. India-rubber loop.
S. First spark-gap.
LL. Lenses.
D. Camera.
F. Tuning-fork and stand.

causes the biggest waves which the above definitions include as ripples; while frequencies of about 200 and upwards give rise to waves whose propagation is practically controlled by surface-tension, and these waves are capillary ripples.

Now the duration of the sensation produced by a luminous impression on the retina lasts for about one eighth of a second; thus we are unable to see ripples on the surface of mercury. The frequencies employed are generally many times the maximum visible frequency. It is not the high velocity of propagation which renders ripples invisible; ripples produced by a disturbance of a frequency of about 200 do not travel very quickly, a foot a second being about the order of magnitude of the velocity on the surface of ordinary mercury.

Description of the Apparatus.

A rectangular wooden trough, about 1 centim. deep and 12 × 15 centim. area, contains the mercury upon the surface of which the ripples are produced. This trough rests upon the rectangular base of a retort-stand, the upper portion of whose standard is bent at right angles over the base so as to be parallel with the latter. This horizontal portion is passed through a rectangular block of wood, and the whole is then slung from a gallows-like wooden structure one metre high.

The spark-gap from which the light proceeds to illuminate the mercury surface is placed near the top and to one side of the wooden stand. Light from this gap falls upon a lens to the left of the stand, so that the emergent light is parallel; after reflexion the light is collected by a second lens similar in all respects to the first, so that an image of the spark would be produced at the primary focus of the second lens; the focal length of these lenses was 44 centim. The camera is placed in such a position as to enable the first achromatic lens of the combination to collect the rays and converge them so as to come to a focus at the aperture in the lens stop. The stop used was the smallest of the set belonging to the camera (F 64). The camera is then focussed upon a fine thread laid on the surface of the mercury.

In addition to the first spark-gap, which was about ·5 centim. across, a second gap was used in order to increase the brightness of the spark in the first gap. The second gap was varied from time to time, but was generally 1·5 centim. across. The first gap was shunted by a piece of stout thread soaked in calcium-chloride solution. This prevented small sparks due to induction. The knobs of a Wimshurst machine were connected, one with a terminal of the second spark-gap and the inside coats of a battery of four half-gallon leyden-jars; the other Wimshurst terminal was connected to the outer coats, one side of the first spark-gap, and to earth. A wire joining the other terminals of the two gaps completes the spark arrangements.

Method of Causing the Ripples.

The ripples are due to the agitation of the surface of the mercury by a style of glass attached to one prong of a vibrating tuning-fork. The fork was in most cases struck with a rubber-shod hammer; but in the last two experiments, in which a strip of cover-glass attached to the fork acts as a line-source, it was found that the irregular large waves caused by the concussion entirely masked the phenomena which it was sought to photograph. In these cases the fork was maintained in synchronous vibration with another similar fork which was electrically excited. The two forks, placed approximately parallel, are tied together by a piece of thread about two feet long, so that the thread is at right angles to both forks. This thread is then adjusted to an appropriate tension by trial, when the maintained fork causes the other to vibrate for any length of time. This simple device, which has been used in the Royal College of Science laboratory for some time, is due to Mr. W. Watson.

Description of Photographs.

The plates used were of various kinds, and any of the well-known plates gave good results. Some of the negatives were intensified previous to printing. The figures are about ½ natural size. My best thanks are due to Mr. R. Chapman, who has assisted me throughout with great zeal.

Photographs showing a series of circular waves set up by a single style attached to a fork of known frequency were taken with a view to quantitative measurements of surface-tension, wave-length, and velocity. A bar of wood was so placed that two needle-points which it carried nearly touched the surface of the mercury. These points are a known distance apart, and by measuring the negatives we may find the scale of reduction of lengths along this line. The particulars of the motion are obtained from the equations

$$v^2 = n^2\lambda^2 = \frac{g\lambda}{2\pi} + \frac{2\pi T}{\lambda\rho};$$

from which we find, in a particular experiment,

$n = 180$ per second,
$\lambda = \cdot 165$ centim.,
$v = 29\cdot 7$ centim. a second,
$T = 306$ dynes per linear centim.

This low value of the surface-tension was obtained from mercury which had stood in the apparatus for some days. The values obtained in a similar way previously were 420, 421, 365. The tension falls as the mercury gets more contaminated. The value of the surface-tension of pure dry mercury is usually quoted at 540.

Fig. 1. Two styles are attached to the same prong of a fork, the frequency of which is 120. One centre is unfortunately hidden by the fork. The approximately straight dark lines which are seen to radiate from the region between the centres of oscillation are lines of minimum disturbance; they are hyperbolas of which the centres of disturbance are the foci. This photograph illustrates the interference phenomena in Optics produced by Young's or Fresnel's methods.

Fig. 2. The frequency of the fork is 256. Both styles are attached to the same prong. The photograph shows two series of interference-curves, one a family of hyperbolas analogous to those shown in fig. 1, and the other a family of ellipses.

The hyperbolas are the radiating light lines seen on the side of the photograph remote from the fork. They are fixed in position, the little dark facets moving along between pairs of hyperbolas.

The light oval curves in the region between the centres of disturbance are ellipses, since they are the loci of points of intersection of the two series of circles whose radii grow uniformly, and at the same rate with time. Their method of production here is similar to a well-known geometrical construction for ellipses. Unlike the system of hyperbolas, these ellipses are not at rest. They travel outwards in such a way that any ellipse occupies a position which was filled previously by its predecessor a whole period before. That semiaxis of any ellipse which passes through a centre of disturbance grows with the same velocity as that with which the ripples are propagated. The other semiaxis grows with a velocity which is infinite at the commencement, but which gradually decreases to the same uniform velocity of growth as that of the first. The law of decrease of velocity is the same as the law of decrease of the lengths of whole-period elements of a linear wave with respect to a point.

In order to render these ellipses stationary it would be necessary to change one of the sources into a sink to which the circular waves converge. This could be experimentally realized with ripples by causing a circular arc and a style to be agitated by the same prong of a fork, when the effects would be analogous to M. Meslin's experiment in Optics.

Fig. 3. Frequency 256.

This photograph is very similar to fig. 1; but in addition to showing interference phenomena like those of Fresnel and Young, it also illustrates interference effects in which the direction of propagation of light is parallel to the line joining the point-sources. Thus in the photograph, if we consider the disturbance anywhere on a right line drawn perpendicular to a line joining the two point-sources produced, we see that the places of no disturbance are symmetrical about the line joining the sources. They are points on the system of hyperbolas already mentioned.

In M. Meslin's method of producing interference-fringes the screen is placed between the two point-centres, one a source and the other a sink. The bands are circular, and are sections of ellipsoids of revolution, and not of hyperboloids, such as the fringes in the photograph would become if the whole picture were rotated about the line joining the point-sources. These fringes are not seen on the screen in M. Meslin's experiment with the split lens, even when the screen is placed beyond the second focus, because the pencils do not there overlap. It seems that modifications of

M. Meslin's experiment could be devised so as to enable complete circular fringes to be seen, and also to render the sections of hyperboloids visible. For example, it appears probable that if a circular portion of a convex lens were cut out and the central portion moved towards the original point-source, the sections of hyperboloids of revolution would be visible on a screen placed beyond the second focus.

Fig. 4. The two sets of ripples are produced by a fork of frequency 128 and another of frequency about 112. These two forks then produce 16 beats a second. The curved light lines represent places of minimum disturbance at the instant when the spark occurred. These lines are not stationary as in No. 3, but rotate towards their convexities. The centre of disturbance from which they move is the one of higher frequency. If we consider a point anywhere on the surface of the mercury, beats occur at that point with the same frequency as the passage of these lines of minimum disturbance takes place over the point. Thus, 16 of these lines cross any point per second.

Fig. 5. This shows ripples produced by two forks, the higher of which has a frequency four times as great as the lower, the frequency of which is 128. If we neglect the effect of gravity,

$$v^2 = n^2\lambda^2 = \frac{2\pi T}{\lambda\rho},$$

from which it follows that the wave-length of the ripples from the higher fork should be half that due to the lower. This relation is approximately true for these ripples.

Fig. 6. Frequency 180.

This photograph shows a point-source and a reflecting-line, the latter is a side of a triangular piece of microscope cover-glass, which is kept in position by a small splinter of wood. The interference-lines which are shown are due to the mutual action of the primary and the reflected waves. The phenomena exhibited are analogous to Lloyd's single-mirror fringes in Optics.

Faint signs of diffraction invading the geometrical shadow of the obstacle can be seen. The region of shadow is covered by faint lines parallel to the nearest side of the triangle acting as a line-source. The wave-length is the same as that of the primary waves, and the effect is due to forced vibrations.

Fig. 7. This photograph illustrates reflexion and forced vibrations. The light curved lines in the region between the source and the nearest side of the triangle are similar to those between the two sources in fig. 2. They are due to the interference between the source and its virtual image.

Fig. 8. Frequency 256.

Here we have a shallow circular reflector with the source placed approximately at the principal focus. The reflected waves are circles of large radii; the very slight outward curvature at the ends shows that even when the reflector has an arc of about 60° the effect of spherical aberration is small. Since the reflected waves come from a virtual point-source, we have, as interference-lines, a series of confocal ellipses and confocal hyperbolas; the latter are fixed, but the former travel away from the line joining the source and its image. If the reflected waves had been rectilinear, both these sets of curves would have become parabolas.

Fig. 9. Frequency 256.

The centre of disturbance here coincides very closely with the principal focus of the central portion of the semicircular reflector. The reflected ripples are straight lines in the middle, but are bent outwards from the reflector towards the ends. This illustrates spherical aberration.

Fig. 10. Frequency 256.

The obstacle is a small round cover-slip floating on the mercury. The ripple-shadow is slightly encroached upon by the waves bending round the edge of the obstacle. One side of the disk acts as a convex circular mirror, and the interference-fringes are due to the mutual action of the source and its virtual image situated within the circumference of the disk.

Fig. 11. The frequency in this and fig. 12 is 120.

Here straight-line waves are originated by the agitation of a slip of cover-glass, one side of which dips into the mercury. The waves are reflected from the shallow circular mirror, and converge to the principal focus. Two series of parabolic interference-fringes are shown. They are confocal, and have their concavities directed towards the source and reflector respectively.

Fig. 12. Similarly excited waves are reflected at an angle of about 45° from a straight edge. The long black mark running from one end of the dipping edge to the corner of the print is due to a depression in the surface caused by a floating needle, put there to screen off the circular waves coming from the end of the strip of glass.

Diffraction is well shown in this photograph.

Discussion.

Mr. Boys congratulated the author upon the way in which the experimental difficulties had been overcome. The results would bear a good deal of close examination, and they would be found to present analogues of the greatest service in demonstrating the phenomena of acoustics and optics. Such photographs were far better than geometrical pictures drawn by instruments. For example, in the photograph illustrating the regions of minimum disturbance by lines radiating from a two-point source, it was easy to make out the positions where the two series of waves were half a period behind one another. The crests and troughs appeared as a set of dark and light concentric alternating circles, broken up into short arcs by radiating lines—the loci of minimum disturbance; all the crests on one side of any particular radiating line were seen to correspond to troughs on the other side, so that the field of disturbance was mapped out as in acoustics. One set of phenomena yet awaited illustration by this photographic method, and that was "diffraction" from a grating. It might be possible to use as an exciter a comb with chisel-shaped points. He did not think it would be possible to go quite so far as to reproduce analogues of spectral analysis. Since wave-length varies with surface tension, it was possible to vary the wave-length by dropping a little ox-gall or soap solution upon the mercury surface.

Mr. Blakesley asked why no reflections occurred from the sides of the mercury retainer.

Mr. Boys said the waves were lost at the edges of the meniscus. The mercury was kept in position by an annular ring of thin glass.

Mr. Appleyard suggested that the analogue of refraction might be obtained by an alteration of the surface-tension over a small area by amalgamation or other means.

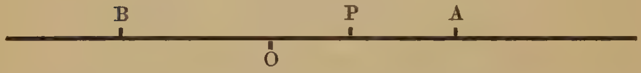
Mr. Vincent thought this could be done, but that it would be very difficult.

IX. *Alternating Currents in Concentric Cables.*

By W. A. Price, *M.A.*[*]

Fig. 1.

§ 1.



AOB is part of an infinite electrical conductor possessing resistance and capacity, the inductance being negligible. The characteristic differential equation is

$$\frac{d^2u}{dx^2}=\rho\gamma\frac{du}{dt},$$

ρ being the resistance, and γ the capacity per unit of length. At A, B are inserted equal alternating electromotive forces in the same phase, each represented by $2\mathrm{E}\sin\omega t$, so that there is no current at O the middle point of AB. The distance AB is 2 L. At any point P distant x from O the potential due to the electromotive force acting at A is

$$\mathrm{E}\,e^{-\alpha(\mathrm{L}-x)}\sin\{\omega t-\alpha(\mathrm{L}-x)\}$$

where

$$\alpha=\sqrt{\frac{\omega\rho\gamma}{2}},$$

and the current at P due to this force acting at A is

$$\frac{\mathrm{E}}{\sqrt{2}}\frac{\alpha}{\rho}e^{-\alpha(\mathrm{L}-x)}[\sin\{\omega t-\alpha(\mathrm{L}-x)\}+\cos\{\omega t-\alpha(\mathrm{L}-x)\}].$$

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[*] Read April 9, 1897.

The current at P due to the electromotive force acting at B is

$$\frac{\mathrm{E}}{\sqrt{2}} \cdot \frac{\alpha}{\rho} e^{-\alpha(\mathrm{L}+x)} [\sin\{\omega t - a(\mathrm{L}+x)\} + \cos\{\omega t - \alpha(\mathrm{L}+x)\}]$$

acting in an opposite direction to the first.

The potential at P due to the electromotive force acting at B is

$$\mathrm{E}e^{-\alpha(\mathrm{L}+x)} \sin\{\omega t - \alpha(\mathrm{L}+x)\}.$$

Then the resultant potential at A is

$$\mathrm{E}\{\sin \omega t + e^{-2\alpha\mathrm{L}} \sin(\omega t - 2\alpha\mathrm{L})\};$$

the potential at O is

$$2\mathrm{E}e^{-\alpha\mathrm{L}} \sin(\omega t - \alpha\mathrm{L});$$

the current at A is

$$\frac{\mathrm{E}}{\sqrt{2}} \cdot \frac{\alpha}{\rho} \{1 - 2e^{-2\alpha\mathrm{L}} \cos 2\alpha\mathrm{L} + e^{-4\alpha\mathrm{L}}\}^{\frac{1}{2}} \cos \omega t.$$

Changing the time from which t is reckoned, so that ωt becomes $\omega t + \phi$, where

$$\tan \phi = \frac{1 - e^{-2\alpha\mathrm{L}}(\cos 2\alpha\mathrm{L} + \sin 2\alpha\mathrm{L})}{1 - e^{-2\alpha\mathrm{L}}(\cos 2\alpha\mathrm{L} - \sin 2\alpha\mathrm{L})},$$

and writing

$$\frac{\mathrm{E}}{\sqrt{2}} \cdot \frac{\alpha}{\rho} \{1 - 2e^{-2\alpha\mathrm{L}} \cos 2\alpha\mathrm{L} + e^{-4\alpha\mathrm{L}}\}^{\frac{1}{2}} \text{ as A, and L}\gamma \text{ as } \Gamma,$$

the current at A becomes $\mathrm{A} \cos \omega t$;
the potential at A becomes

$$\frac{\mathrm{A}}{\omega\Gamma} \alpha\mathrm{L} \frac{(\sinh 2\alpha\mathrm{L} + \sin 2\alpha\mathrm{L}) \sin \omega t + (\sinh 2\alpha\mathrm{L} - \sin 2\alpha\mathrm{L}) \cos \omega t}{\cosh 2\alpha\mathrm{L} - \cos 2\alpha\mathrm{L}};$$

and the potential at O becomes

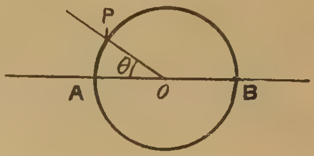
$$\frac{2\mathrm{A}}{\omega\Gamma} \alpha\mathrm{L} \frac{(\cos a\mathrm{L} \sinh a\mathrm{L} + \sin a\mathrm{L} \cosh a\mathrm{L}) \sin \omega t + (\cos a\mathrm{L} \sinh a\mathrm{L} - \sin a\mathrm{L} \cosh a\mathrm{L}) \cos \omega t}{\cosh 2a\mathrm{L} - \cos 2a\mathrm{L}}.$$

The form and notation of these expressions are those given in 'Alternating Currents of Electricity,' T. H. Blakesley, London, 1889.

Consider the circular electrical conductor AB (fig. 2), having a total resistance $2\pi r$ and a total capacity $2\pi c$: the characteristic equation being

$$\frac{d^2u}{d\theta^2}=rc\,.\,\frac{du}{dt}.$$

Fig. 2.



At a time $t=0$, when the conductor is uncharged, a charge $2\pi\mathrm{S}$ is introduced at A. The charge at any point P at a subsequent time t is

$$s=\mathrm{S}\sum_{-\infty}^{\infty}\cos i\theta e^{-\mu i^2 t},\text{ where }\mu=\frac{1}{rc}:$$

$s\,d\theta$ being the charge at P, where $\mathrm{AOP}=\theta$, of a small arc subtending $d\theta$ at the centre.

If a series of charges, $2\pi\mathrm{S}\cos\omega\tau\,.\,d\tau$, is introduced at A at intervals of time $d\tau$ beginning when $\tau=0$, s, the charge at P at a time t,

$$=\mathrm{S}\sum_{-\infty}^{\infty}\cos i\theta\left\{\frac{\mu i^2\cos\omega t+\omega\sin\omega t}{\omega^2+\mu^2 i^4}-\frac{\mu i^2}{\omega^2+\mu^2 i^4}e^{-\mu i^2 t}\right\}.$$

If this succession of charges equivalent to a permanent alternating current $2\pi\mathrm{S}\cos\omega t$ be continued for a long time, the term involving $e^{-\mu i^2}$ becomes very small and we have

$$s=\mathrm{S}\sum_{-\infty}^{\infty}\cos i\theta\frac{\mu i^2\cos\omega t+\omega\sin\omega t}{\omega^2+\mu^2 i^4}.$$

The potential at A is

$$\frac{\mathrm{S}}{\omega c}\sin\omega t\sum_{-\infty}^{\infty}\left(1+\frac{\mu^2 i^4}{\omega^2}\right)^{-1}+\frac{\mathrm{S}}{\omega c}\cos\omega t\sum_{-\infty}^{\infty}\frac{\mu i^2}{\omega}\left(1+\frac{\mu^2 i^4}{\omega^2}\right)^{-1};$$

the potential at B is

$$\frac{\mathrm{S}}{\omega c}\sin\omega t\sum_{-\infty}^{\infty}(-1)^i\left(1+\frac{\mu^2 i^4}{\omega^2}\right)^{-1}+\frac{\mathrm{S}}{\omega c}\cos\omega t\sum_{-\infty}^{\infty}(-1)^i\frac{\mu i^2}{\omega}\left(1+\frac{\mu^2 i^4}{\omega^2}\right)^{-1}.$$

It may be seen that the electrical actions in the two cases discussed are identical so far as the portion AB of the conductor in the first case is concerned.

Writing $L\gamma=\pi c$, $L\rho=\pi r$, $\pi S=A$, and $\sqrt{\frac{\omega}{2\mu}}$ as β, so that $\alpha L=\beta\pi$,

$$\sum_{-\infty}^{\infty}\left(1+\frac{i^4}{4\beta^4}\right)^{-1}=\beta\pi\frac{\sinh 2\beta\pi+\sin 2\beta\pi}{\cosh 2\beta\pi-\cos 2\beta\pi}$$

[this expression I shall refer to as $f_1(\beta)$];

$$\sum_{-\infty}^{\infty}\frac{i^2}{2\beta^2}\left(1+\frac{i^4}{4\beta^4}\right)^{-1}=\beta\pi\frac{\sinh 2\beta\pi-\sin 2\beta\pi}{\cosh 2\beta\pi-\cos 2\beta\pi}$$

[this expression I shall refer to as $f_2(\beta)$];

$$\sum_{-\infty}^{\infty}(-1)^i\left(1+\frac{i^4}{4\beta^4}\right)^{-1}=2\beta\pi\frac{\cos\beta\pi\sinh\beta\pi+\sin\beta\pi\cosh\beta\pi}{\cosh 2\beta\pi-\cos 2\beta\pi}$$

[this expression I shall refer to as $f_3(\beta)$];

$$\sum_{-\infty}^{\infty}(-1)^i\frac{i^2}{2\beta^2}\left(1+\frac{i^4}{4\beta^4}\right)^{-1}=2\beta\pi\frac{\cos\beta\pi\sinh\beta\pi-\sin\beta\pi\cosh\beta\pi}{\cosh 2\beta\pi-\cos 2\beta\pi}$$

[this expression I shall refer to as $f_4(\beta)$].

In the particular case where β is integral

$$\sum_{-\infty}^{\infty}\left(1+\frac{i^4}{4\beta^4}\right)^{-1}=\sum_{-\infty}^{\infty}\frac{i^2}{2\beta^2}\left(1+\frac{i^4}{4\beta^4}\right)^{-1}=\beta\pi\coth\beta\pi,$$

$$\sum_{-\infty}^{\infty}(-1)^i\left(1+\frac{i^4}{4\beta^4}\right)^{-1}=\sum_{-\infty}^{\infty}(-1)^i\frac{i^2}{2\beta^2}\left(1+\frac{i^4}{4\beta^4}\right)^{-1}=\beta\pi\,\mathrm{cosech}\,\beta\pi.$$

These last expressions may also be obtained by expanding $(\cos\beta\theta\pm\sin\beta\theta)e^{\beta\theta}$ in Fourier series of cosines.

For

$$\beta\pi(\cos\beta\theta-\sin\beta\theta)e^{\beta\theta}$$
$$=\sum_{-\infty}^{\infty}(-1)^i\{(-1)^\beta e^{\beta\pi}-(-1)^i\}\left\{1+\frac{i^4}{4\beta^4}\right\}^{-1}\cos i\theta,$$

$$\beta\pi(\cos\beta\theta+\sin\beta\theta)e^{\beta\theta}$$
$$=\sum_{-\infty}^{\infty}(-1)^i\{(-1)^\beta e^{\beta\pi}-(-1)^i\}\frac{i^2}{2\beta^2}\left\{1+\frac{i^4}{4\beta^4}\right\}^{-1}\cos i\theta.$$

Putting θ equal to 0 and π successively in these series, and adding and subtracting the series thus obtained in pairs, the four last expressions are obtained. The method is applicable only when β is integral.

In a paper by Dr. Glaisher ('Proc. London Math. Soc.,' vol. vii.) the value of $\sum\limits_{-\infty}^{\infty}\frac{1}{r^4+x^4}$ is determined by another method which gives the same result as the above.

The physical significance of β being integral is that in that case the value of ω is such that the alternations at the sending and receiving ends of the cable are in the same phase. The charge at the sending end being

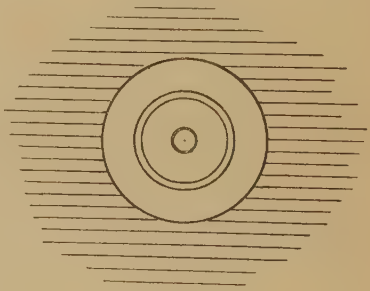
$$s=\frac{\mathrm{S}}{\omega}\beta\pi\cosh\beta\pi(\sin\omega t+\cos\omega t)$$

and at the receiving end

$$s=\frac{\mathrm{S}}{\omega}\beta\pi\,\mathrm{cosech}\,\beta\pi(\sin\omega t+\cos\omega t),$$

the ratio of the amplitudes being $\cosh\beta\pi$.

Fig. 3.



§ 2. A cable, shown in section in fig. 3, contains two conductors, an inner central conductor and an outer concentric conductor, insulated from one another and the water in which the whole is immersed. The whole cable is supposed to be arranged in the circular form of fig. 2.

$2\pi r$, $2\pi r'$ are the total resistances of the inner and outer conductors ;

$2\pi m$ the total capacity of the inner dielectric measured between the two conductors ;

$2\pi n$ the total capacity of the outer dielectric measured between the outer conductor and the water.

Then $2\pi\frac{mn}{m+n}$ is the capacity of the central conductor, the outer one being insulated; and $2\pi n$ the capacity of the outer conductor, the inner one being insulated.

$u\ u'$, C C′ are the potentials of and the currents in the two conductors at point P and time t, and $s\ s'$ the density of the charges on the conductors. All these quantities being referred to a unit of length subtending unit angle at the centre, then

$$rC=-\frac{du}{d\theta}, \qquad r'C'=-\frac{du'}{d\theta};$$

$$-\frac{dC}{d\theta}=m\left(\frac{du}{dt}-\frac{du'}{dt}\right), \quad -\frac{dC'}{d\theta}=m\left(\frac{du'}{dt}-\frac{du}{dt}\right)+n\frac{du}{dt};$$

$$s+s'=nu', \qquad \text{and} \quad s=m(u-u').$$

Also
$$\left(\frac{d^2}{d\theta^2}-mr\frac{d}{dt}\right)s+mr'\frac{d}{dt}s'=0,$$

and
$$\left(\frac{d^2}{d\theta^2}-(m+n)r'\frac{d}{dt}\right)s'+mr\frac{d}{dt}s=0.$$

Hence s and s' satisfy the equation

$$\left(\mu\frac{d^2}{d\theta^2}-\frac{d}{dt}\right)\left(\nu\frac{d^2}{d\theta^2}-\frac{d}{dt}\right)s=0,$$

where μ, ν are roots of $mnrr'x^2-(mr+(m+n)r')x+1=0$.

§3. A uniform conductor Q, along which the propagation of charges due to any electrical disturbance is determined by the differential equation

$$\left(\mu\frac{d^2}{d\theta^2}-\frac{d}{dt}\right)\left(\nu\frac{d^2}{d\theta^2}-\frac{d}{dt}\right)s=0,$$

is arranged in a circle as in fig. 2. Its total resistance is $2\pi r$, and its total capacity $2\pi c$. Any disturbance will be propagated along the conductor in two distinct systems, one determined by the differential equation

$$\left(\mu\frac{d^2}{d\theta^2}-\frac{d}{dt}\right)s=0, \qquad \text{the other by} \left(\nu\frac{d^2}{d\theta^2}-\frac{d}{dt}\right)s=0;$$

and the disturbance produced at any given point will be the sum of two disturbances conveyed by the two systems.

Since each propagation is determined by a differential equation of the same form as that which applies to a single isolated conductor, such as is discussed in § 1, the action of Q is equivalent to that of a pair of single conductors, and the propagation along Q is the sum of the propagations along the equivalent pair.

Let the total resistances of the two equivalent cables be $2\pi r_1$ and $2\pi r_2$, and their capacities $2\pi c_1$ $2\pi c_2$. Then $\mu r_1 c_1 = 1$ and $\nu r_2 c_2 = 1$.

Since the pair is electrically equivalent to Q, they must convey the same steady current as Q, when subject to a steady E.M.F., and must be charged to the same potential by a given charge. So $r_1^{-1} + r_2^{-1} = r^{-1}$, and $c_1 + c_2 = c$. From these four equations r_1 r_2 c_1 c_2 can be determined.

Then if a charge $2\pi S$ be introduced into this conductor at any point, the charge at any point after a time t will be the sum of the charges at the corresponding points of the equivalent pair, if the same charge had been introduced into them in the proportions $2\pi S\frac{c_1}{c}$, $2\pi S\frac{c_2}{c}$.

If an alternating current $2\pi S \cos \omega t$ be introduced into the conductor at A where $\theta = 0$, the charge at P will be

$$s = \frac{S}{\omega}\frac{c_1}{c}\sum_{-\infty}^{\infty} \cos i\theta \frac{\mu i^2 \cos \omega t + \omega \sin \omega t}{\omega^2 + \mu^2 i^4}$$

$$+ \frac{S}{\omega}\frac{c_2}{c}\sum_{-\infty}^{\infty} \cos i\theta \frac{\nu i^2 \cos \omega t + \omega \sin \omega t}{\omega^2 + \nu^2 i^4} = s_1 + s_2.$$

§ 4. Applying this argument to the conductors of the cable described in § 2, let r_1 r_2 c_1 c_2 be the resistances and capacities of the pair equivalent to the central conductor, and r_1' r_2' c_1' c_2' of the pair equivalent to the concentric; so that

$$r_1^{-1} + r_2^{-1} = r^{-1}, \qquad r_1'^{-1} + r_2'^{-1} = r'^{-1};$$

$$r_1 c_1 \mu = r_2 c_2 \nu = r_1' c_1' \mu = r_2' c_2' \nu = 1;$$

$$c_1 + c_2 = \frac{mn}{m+n} = c, \qquad c_1' + c_2' = c' = n.$$

Then
$$\frac{c_1}{c} = \frac{c_2'}{c'} = \frac{\mu}{\mu - \nu}(1 - m\nu r),$$

and
$$\frac{c_2}{c} = \frac{c_1'}{c'} = \frac{\nu}{\nu - \mu}(1 - m\mu r).$$

Then the charge at any point P due to an alternating current $2\pi S \cos \omega t$ introduced at A, where $\theta = 0$, is $s = s_1 + s_2$,

where

$$s_1 = \mathrm{S}\frac{\mu}{\mu-\nu}(1-m\mu r)\sum_{-\infty}^{\infty}\cos i\theta\frac{\mu i^2\cos\omega t+\omega\sin\omega t}{\omega^2+\mu^2 i^4},$$

$$s_2 = \mathrm{S}\frac{\nu}{\nu-\mu}(1-m\nu r)\sum_{-\infty}^{\infty}\cos i\theta\frac{\nu i^2\cos\omega t+\omega\sin\omega t}{\omega^2+\nu^2 i^4}.$$

§ 5. If an alternating current $2\pi\mathrm{S}\cos\omega t$ be introduced into the concentric conductor at A, the charge s' at any point P $=s_1'+s_2'$, where

$$s_1' = \frac{\mathrm{S}}{\omega}\frac{\nu(1-\mu mr)}{\nu-\mu}\sum_{-\infty}^{\infty}\cos i\theta\frac{\mu i^2\cos\omega t+\omega\sin\omega t}{\omega^2+\mu^2 i^4},$$

$$s_2' = \frac{\mathrm{S}}{\omega}\frac{\mu(1-\nu mr)}{\mu-\nu}\sum_{-\infty}^{\infty}\cos i\theta\frac{\nu i^2\cos\omega t+\omega\sin\omega t}{\omega^2+\nu^2 i^4}.$$

§ 6. Consider the charge s' induced in the concentric conductor by the charge s_1+s_2 in the central conductor.

From § 2,

$$\left(\frac{d^2}{d\theta^2}-mr\frac{d}{dt}\right)(s_1+s_2)+mr'\frac{d}{dt}s'=0,$$

$$\frac{d^2}{d\theta^2}s_1=\frac{1}{\mu}\frac{d}{dt}s_1, \qquad \frac{d^2}{d\theta^2}s_2=\frac{1}{\nu}\frac{d}{dt}s_2;$$

hence $$\frac{1-m\mu r}{\mu}\frac{ds_1}{dt}+\frac{1-m\nu r}{\nu}\frac{ds_2}{dt}+mr'\frac{ds'}{dt}=0.$$

Integrating, $mr's'=-\frac{1}{\mu}(1-m\mu r)s_1-\frac{1}{\nu}(1-m\nu r)s_2+$ some expression Θ independent of t.

Θ is periodic in θ, satisfies $\left(\mu\frac{d^2}{d\theta^2}-\frac{d}{dt}\right)\left(\nu\frac{d^2}{d\theta^2}-\frac{d}{dt}\right)\Theta=0$, and has a mean value zero. Hence $\Theta=0$, and

$$mr's'=-\frac{1}{\mu}(1-m\mu r)s_1-\frac{1}{\nu}(1-m\nu r)s_2.$$

In the same way the charge s induced in the central conductor by the distribution $s_1'+s_2'$ in the concentric conductor is given by the equations

$$\left\{\frac{d^2}{d\theta^2}-(m+n)r'\frac{d}{dt}\right\}(s_1'+s_2')+mr\frac{d}{dt}s=0,$$

$$\frac{d^2}{d\theta^2}s_1'=\frac{1}{\mu}\frac{d}{dt}s_1', \qquad \frac{d^2}{d\theta^2}s_2'=\frac{1}{\nu}\frac{d}{dt}s_2',$$

and

$$mrs=\frac{1}{\nu}(1-mr\nu)s_1'+\frac{1}{\mu}(1-mr\mu)s_2'.$$

§ 7. Suppose an alternating current $2\pi A\cos\omega t$ to be introduced into the central conductor at A where $\theta=0$, and a current $2\pi B\cos(\omega t+\phi)$ into the same conductor at B where $\theta=\pi$.

The charge at B on the central conductor due to the current introduced at A is, writing

$$\sqrt{\frac{\omega}{2\mu}}\text{ as }\beta,\text{ and }\sqrt{\frac{\omega}{2\nu}}\text{ as }\gamma,$$

$$\frac{A}{\omega}\frac{c_1}{c}\{\sin\omega t\,f_3(\beta)+\cos\omega t\,f_4(\beta)\}$$
$$+\frac{A}{\omega}\frac{c_2}{c}\{\sin\omega t\,f_3(\gamma)+\cos\omega t\,f_4(\gamma)\}.$$

The charge at B on the central conductor due to the current introduced at B is

$$\frac{B}{\omega}\frac{c_1}{c}\{\sin(\omega t+\phi)f_1(\beta)+\cos(\omega t+\phi)f_2(\beta)\}$$
$$+\frac{B}{\omega}\frac{c_2}{c}\{\sin(\omega t+\phi)f_1(\gamma)+\cos(\omega t+\phi)f_2(\gamma)\}.$$

The charge at B induced on the concentric conductor by the current introduced at A is

$$-\frac{1}{m\mu r'}(1-m\mu r)\frac{A}{\omega}\frac{c_1}{c}\{\sin\omega t\,f_3(\beta)+\cos\omega t\,f_4(\beta)\}$$
$$-\frac{1}{m\nu r'}(1-m\nu r)\frac{A}{\omega}\frac{c_2}{c}\{\sin\omega t\,f_3(\gamma)+\cos\omega t\,f_4(\gamma)\}.$$

The charge at B induced on the concentric conductor by the current introduced at B is

$$-\frac{1}{m\mu r'}(1-m\mu r)\frac{\mathrm{B}}{\omega}\frac{c_1}{c}\{\sin(\omega t+\phi)f_1(\beta)+\cos(\omega t+\phi)f_2(\beta)\}$$

$$-\frac{1}{m\nu r'}(1-m\nu r)\frac{\mathrm{B}}{\omega}\frac{c_2}{c}\{\sin(\omega t+\phi)f_1(\gamma)+\cos(\omega t+\phi)f_2(\gamma)\}.$$

The potential of the central conductor at the point B is $\frac{(m+n)s+ms'}{mn}$, where $s\,s'$ are the total charges on the two conductors respectively, and is

$$\mu r\frac{\mathrm{A}}{\omega}\frac{c_1}{c}\{\sin\omega t\,f_3(\beta)+\cos\omega t\,f_4(\beta)\}$$

$$+\nu r\frac{\mathrm{A}}{\omega}\frac{c_2}{c}\{\sin\omega t\,f_3(\gamma)+\cos\omega t\,f_4(\gamma)\},$$

$$+\mu r\frac{\mathrm{B}}{\omega}\frac{c_1}{c}\{\sin(\omega t+\phi)f_1(\beta)+\cos(\omega t+\phi)f_2(\beta)\}$$

$$+\nu r\frac{\mathrm{B}}{\omega}\frac{c_2}{c}\{\sin(\omega t+\phi)f_1(\gamma)+\cos(\omega t+\phi)f_2(\gamma)\}.$$

If a known current $2\pi\mathrm{A}\cos\omega t$ be introduced at A, and the point B be kept at zero potential, the equation of the last expression to zero for every value of t gives two equations to determine B and ϕ, the amplitude and phase of the current at the receiving end of the cable. These are

$$\mathrm{A}\{\mu c_1 f_3(\beta)+\nu c_2 f_3(\gamma)\}+\mathrm{B}[\mu c_1\{\cos\phi\,f_1(\beta)-\sin\phi\,f_2(\beta)\}$$
$$+\nu c_2\{\cos\phi\,f_1(\gamma)-\sin\phi\,f_2(\gamma)\}]=0;$$

$$\mathrm{A}\{\mu c_1 f_4(\beta)+\nu c_2 f_4(\gamma)\}+\mathrm{B}[\mu c_1\{\sin\phi\,f_1(\beta)+\cos\phi\,f_2(\beta)\}$$
$$+\nu c_2\{\sin\phi\,f_1(\gamma)+\cos\phi\,f_2(\gamma)\}]=0.$$

Similarly, if the currents had been introduced into the concentric conductor, the central one being insulated throughout, we should have for the potential of the concentric conductor at B, $\frac{1}{n}(s+s')$, which is exactly the same expression as for the central conductor substituting r' for r, $\frac{c_1'}{c}$ for $\frac{c_1}{c}$, and

$\frac{c_2'}{c}$ for $\frac{c_2}{c}$. Remembering that $\frac{c_1}{c}=\frac{c_2'}{c'}$ and $\frac{c_2}{c}=\frac{c_1'}{c''}$, it is seen that the equations for determining the phase and amplitude of the current received at B when that end of the concentric conductor is earthed are the same as in the case of the central conductor, only interchanging c_1 and c_2.

§ 8. So if an alternating current, $2\pi\text{A} \cos(\omega t + a)$, be introduced into the central conductor at A, and a current $2\pi\text{B} \cos(\omega t+b)$ pass out of the concentric conductor at B, the potentials of the conductors at A, B may be determined.

The potential of the central conductor at A is

$$\frac{1}{mn}\{(m+n)s+ms'\}$$

where s s' are the charges on the two conductors.

$$\begin{aligned}
\text{This} =& \frac{\text{A}}{\omega}\cdot\frac{\mu^2 r}{\mu-\nu}(1-m\nu r)\{\sin(\omega t+a)f_1(\beta)+\cos(\omega t+a)f_2(\beta)\} \\
&+\frac{\text{A}}{\omega}\cdot\frac{\nu^2 r}{\nu-\mu}(1-m\mu r)\{\sin(\omega t+a)f_1(\gamma)+\cos(\omega t+a)f_2(\gamma)\} \\
&-\frac{\text{B}}{\omega}\cdot\frac{\mu}{n(\mu-\nu)}\{\sin(\omega t+b)f_3(\beta)+\cos(\omega t+b)f_4(\beta)\} \\
&-\frac{\text{B}}{\omega}\cdot\frac{\nu}{(\nu-\mu)n}\cdot\{\sin(\omega t+b)f_3(\gamma)+\cos(\omega t+b)f_4(\gamma)\}.
\end{aligned}$$

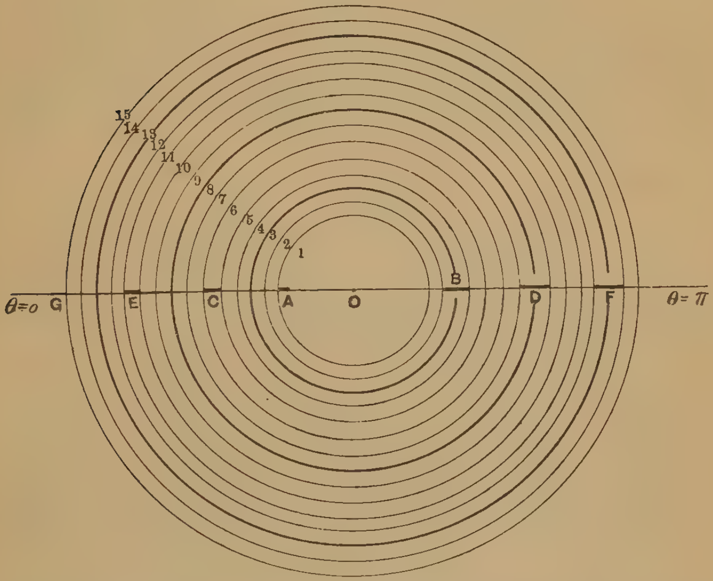
The potential of the concentric conductor at B is $\frac{1}{n}(s+s')$.

$$\begin{aligned}
\text{This} =& \frac{\text{A}}{\omega}\cdot\frac{\mu}{n(\mu-\nu)}\{\sin(\omega t+a)f_3(\beta)+\cos(\omega t+a)f_4(\beta)\} \\
&+\frac{\text{A}}{\omega}\cdot\frac{\nu}{n(\nu-\mu)}\{\sin(\omega t+a)f_3(\gamma)+\cos(\omega t+a)f_4(\gamma)\} \\
&+\frac{\text{B}}{\omega}\cdot\frac{\mu\nu r'}{\mu-\nu}(1-m\mu r)\{\sin(\omega t+b)f_1(\beta)+\cos(\omega t+b)f_2(\beta)\} \\
&+\frac{\text{B}}{\omega}\cdot\frac{\mu\nu r'}{\nu-\mu}(1-m\nu r)\{\sin(\omega t+b)f_1(\gamma)+\cos(\omega t+b)f_2(\gamma)\}.
\end{aligned}$$

§ 9. Consider a system of circular conductors, of which fifteen are shown in fig. 4.

Of these Nos. 1, 5, 6, 10, 11, 15 have each a resistance $2\pi r$; they represent the central conductors of such concentric cables as we have been considering. Nos. 2, 4, 7, 9, 12, 14 have each a resistance $2\pi r'$. They represent the outer or concentric conductors. Nos. 3, 8, 13, have no resistance, and are maintained always at zero potential throughout. They represent the water or sheathing of the cables.

Fig. 4.



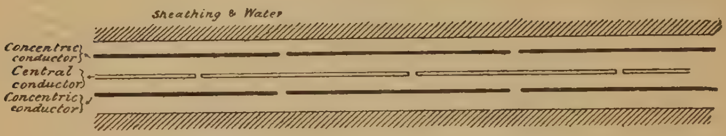
The capacity measured between each of the pairs 1, 2 : 4, 5 : 6, 7 : 9, 10 : 11, 12 : 14, 15 respectively is $2\pi c$, where $c = m$; and between the pairs 2, 3 : 3, 4 : 7, 8 : 8, 9 : 12, 13 : 13, 14 is $2\pi c'$, where $c' = n$. The capacity between the pairs 5, 6 : 10, 11 is zero.

At B conductors 2, 4 are connected by a piece having no resistance or capacity.

Similarly at C, conductors 5, 6 are connected; at D, 7, 9; at E, 10, 11; at F, 12, 14, and so on. The series may be indefinitely extended.

The system is electrically equivalent to a length of concentric cable of which the two conductors are cut alternately at equidistant points in the way shown in fig. 5, representing a longitudinal section of the cable. If an alternating current be introduced into the system of fig. 4 at any link, say E, and the point A be kept at zero potential, alternating currents of the same period, but of different phases and amplitudes, will be found at every point of both conductors.

Fig. 5.



Suppose the currents at the points A, B, C, D, . . G to be

$$A\cos(\omega t+a),\ B\cos(\omega t+b),\ .\ .\ G\,(\cos\omega t+g),$$

so that the values of Gg being known, and the point A being kept at zero potential, the values of Aa, Bb, Cc, . . Ff are to be determined, and especially the values of Aa.

The potential at the point C on the conductor 5 has been expressed in terms of Cc, Bb ; and the potential at the point C on the conductor 6 in terms of Cc, Dd. These two are equal for all values of t, and we have an equation which can be separated into two others between Bb, Cc, Dd. In the same way we obtain two equations at A, B, D, E, and F, and between these all the quantities required can be obtained.

In the particular case where ω has such a value that β and γ are integral, and the currents at all the points are in the same phase, let us write in § 8,

$$\frac{\mu^2 r}{\mu-\nu}(1-m\nu r)f_1(\beta)+\frac{\nu^2 r}{\nu-\mu}(1-m\mu r)f_1(\gamma)\ \text{as P},$$

$$\frac{\mu}{n(\mu-\nu)}f_3(\beta)+\frac{\nu}{n(\nu-\mu)}f_3(\gamma)\ \text{as Q},$$

$$\frac{\mu\nu r'}{\nu-\mu}(1-m\mu r)f_1(\beta)+\frac{\mu\nu r'}{\mu-\nu}(1-m\nu r)f_1(\gamma)\ \text{as R},$$

remembering that when β γ are integral, $f_1(\beta)=f_2(\beta)$, and $f_3(\beta)=f_4(\beta)$, and the same with γ.

The condition at A gives $AP = BQ$,

" " B " $(A+C)Q = 2BR$,

" " C " $(B+D)Q = 2CP$,

" " D " $(C+E)Q = 2DR$, and so on.

If the known current is introduced at B, and A is kept to earth, the ratio of the amplitude of the received current to the sent current is given by

$$\frac{B}{A} = \frac{P}{Q} \quad . \; . \; . \; . \; . \; . \quad \text{Case I.}$$

If the known current is introduced at C, we have

$$\frac{C}{A} = 2\frac{PR}{Q^2} - 1 \quad . \; . \; . \; . \quad \text{Case II.}$$

If the known current is introduced at E, we have

$$\frac{E}{A} = 8\left(\frac{PR}{Q^2}\right)^2 - 8\frac{PR}{Q^2} + 1 \quad . \; . \quad \text{Case III.}$$

If the known current is introduced at G, we have

$$\frac{G}{A} = 32\left(\frac{PR}{Q^2}\right)^3 - 48\left(\frac{PR}{Q^2}\right)^2 + 18\frac{PR}{Q^2} - 1 \quad . \quad \text{Case IV.}$$

§ 10. Applying these results to a cable having the following constants, represented by half the circle of fig. 4:

Length 4000 kilometres $= 4.10^8$ units.

Resistance of each conductor, 5 ohms per kilometre, *i. e.* 2.10^{13} units for the whole cable.

Capacity measured between the conductors, $\frac{5+\sqrt{7}}{18}10^{-15}$ units per kilometre (about ·425 mfds.), *i. e.* $\frac{5+\sqrt{7}}{18}4.10^{-12}$ units for the whole cable.

Capacity measured between the outer conductor and the sheathing, $\frac{5-\sqrt{7}}{9}10^{-15}$ units per kilometre (about ·26 mfds.), *i. e.* $\frac{5-\sqrt{7}}{9}.4.10^{-12}$ units for the whole cable.

Then

$$m=\frac{2}{\pi}\frac{5+\sqrt{7}}{9}10^{-12}, \qquad n=\frac{4}{\pi}\frac{5-\sqrt{7}}{9}10^{-12},$$

$$r=r'=\frac{2}{\pi}10^{13}, \qquad \mu=\frac{\pi^2}{80}, \qquad \nu=\frac{9\pi^2}{80},$$

$$\frac{c_1}{c}=\frac{c_2'}{c'}=\frac{3+\sqrt{7}}{16}, \qquad \frac{c_2}{c}=\frac{c_1'}{c'}=\frac{13-\sqrt{7}}{16}.$$

Consider the cable in four cases :—

Case I. when both conductors are continuous.

Case II. when the inner conductor is cut in the middle so that the cable is in two sections.

Case III. when the inner conductor is cut in two places, and the outer in one, so that the cable is in four sections.

Case IV. when the inner conductor is cut in three places, and the outer in two, so that the cable is in six sections.

Take a value of $\omega\,.\,\frac{9\,.\,144\,.\,\pi^2}{80}$ which makes $\beta\;\;\gamma$ integral in all cases. This corresponds to about 25 alternations per second.

Then if the cable be divided into n equal sections the new values of $rr'mn\beta\gamma$ are inversely proportional to n, while the values of μ, ν are proportional to the square of n. The value of $\frac{\mathrm{PR}}{\mathrm{Q}^2}$ is unchanged in every case.

In Case I. $\beta=30\;\gamma=12$, and $\mathrm{A}=1{\cdot}26\,e^{-12\pi}\mathrm{B}$;
Case II. $\beta=18\;\gamma=6$, and $\mathrm{A}=1{\cdot}07\,e^{-12\pi}\mathrm{C}$;
Case III. $\beta=9\;\gamma=3$, and $\mathrm{A}={\cdot}57\,e^{-12\pi}\mathrm{E}$;
Case IV. $\beta=6\;\gamma=2$, and $\mathrm{A}={\cdot}31\,e^{-12\pi}\mathrm{G}$.

Thus the effect of each successive subdivision is to further reduce the amplitude of the received current.

§11. Returning to the equation of § 2 for determining $\mu\nu$, viz. :—

$$mnrr'x^2-x\{\overline{m+n}\nu'+m\nu\}+1=0,$$

write in this

$$\frac{1}{r}=\gamma,\ \ \frac{1}{r'}=\gamma',\ \ \frac{1}{m}=k,\ \ \frac{1}{n}=k',\ \text{ and } \gamma+\gamma'=\Gamma,$$

so that γ, γ' are the conductivities of the two conductors, k, k' the rigidities to electrostatic stress of the two dielectrics, and Γ the total conductivity of the two conductors.

The equation becomes

$$x^2-x\{\gamma k+\Gamma k'\}+\gamma\gamma' kk'=0,$$

and the roots of this equation represent the signalling speeds of the two equivalent cables.

If k be very small, $x=\Gamma k'$ or 0. Hence if the layer of dielectric between the conductors be very thin, though at the same time perfectly insulating, the speed through the central conductor, however small its section may be, is the same as if the whole of the two conductors were solid, and the whole used for the conducting circuit.

Discussion.

Mr. Blakesley said he was sorry the result did not indicate a successful type of cable. He would have been inclined to predict that the amplitude would have decreased with the number of sections. If a number of condensers were joined in series, and one end was subjected to a periodic E.M.F., the amplitude would fall off inversely as the square of the number of the condensers, the total capacities remaining the same.

Mr. Price then exhibited a galvanometer support. The instrument is suspended from two indiarubber cords attached at the top and bottom to cross-bars of metal, thus forming a rectangle. The cross-bars are provided with knife-edges in such a way as to compensate for unequal stretching of the indiarubber. Weights can be added, if necessary, to the support, so as to increase its inertia.

X. *A Nickel Stress Telephone.* *By* T. A. Garrett, *M.A.*, *and* William Lucas, *M.A.*[*]

It is well known that, if a magnetized nickel wire be subjected to longitudinal stress, variations of this stress produce variations in the magnetization of the wire. If the stress be a pressure, an increase in the pressure produces an increase in the magnetization, and a decrease in the pressure a decrease in the magnetization.

Suppose then that a magnetized piece of nickel wire, round which a coil of insulated wire has been wound, is fixed in a vertical position by a clamp at its lower end, and that its upper end is fastened to the centre of a light horizontal plate, which serves as a diaphragm. On speaking against the top of the diaphragm variations of longitudinal pressure and consequently of magnetization will be produced in the nickel wire, and hence a varying current will be induced in the coil. If the coil be connected with any kind of receiving telephone the speech will be reproduced.

We have made a large number of experiments with the object of determining the form of instrument which would give the best results; but owing to the difficulties of comparing the speaking powers of different instruments we found it impossible to say with certainty which form was the best. The details of one of the forms with which good results were obtained are as follows:—The nickel wire was 10 cm. long and 1 mm. in diameter. The clamp was formed of a massive piece of gunmetal (an ordinary workshop vice will do quite as well) and the diaphragm was a circular disk of pine wood 3·5 mm. thick and 12·5 cm. in diameter. A hole was made through it in the centre into which the nickel wire was stuck with sealing-wax. This diaphragm was entirely supported by the nickel wire. The coil consisted of 3·9 grms. of No. 40 B.W.G. silk-covered wire wound directly on the nickel wire. The length of the coil was about 7·5 cm. An ordinary watch telephone, the resistance of the coils of which was about 136 ohms, was used as a receiver. The resistance of the line was quite small, as all the experiments were made from one room to another in the same building. The nickel wire was sometimes magnetized by stroking it with a magnet, sometimes by sending a current from a few Leclanché cells through the coil. Whether the wire was annealed or not appeared to be immaterial. Wires of various dimensions were tried, but the results obtained were not very different. With a diaphragm made of pine wood, the results were decidedly better than with a metallic diaphragm articulation then being very clear. It was found to be better to use a clamp of considerable mass.

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[*] Read April 9, 1897.

When the nickel wire was replaced by an iron wire the results obtained were very feeble.

The same instrument was also used as a receiver. With an iron wire it worked very badly, with a nickel wire it worked rather better but not at all well. Since beginning our experiments we have found that M. Ader* has used as a receiver an instrument somewhat similar to the above, except that the wire was of iron instead of nickel.

In order to show that the current induced in the coil is at any rate chiefly due to variations in the magnetization of the nickel wire produced by the variations of stress, and not to the relative motion of the nickel wire and the coil, the following experiment was made :—A steel wire and a nickel wire of the same dimensions were attached to similar wooden diaphragms. These wires having been magnetized by stroking them with a permanent magnet were in turn inserted into the same solenoid and clamped as before at their lower ends. The same watch telephone was used as a receiver with each. The results obtained with the weakly magnetized nickel wire were enormously better than those obtained with the strongly magnetized steel wire. If the induced currents were chiefly due to the relative motion of the coil and magnetized wire the best results would have been obtained with the strongly magnetized steel wire, since it can hardly be supposed that the relative motions of the coil and magnetized wire differed so much in the two cases as to cause such an enormous difference in the results.

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* *Comp. Rend.* 1879, t. lxxxviii. p. 575

Discussion.

Dr. S. P. Thompson said that some years ago he had worked with a somewhat similar apparatus, using it as a "receiver," with wires of nickel, cobalt, and iron. Cobalt gave the best results ; the metallic strips in his experiments dipped into the solenoids without contact with them. This arrangement did not work well as a "transmitter," even when a battery was included in the circuit. In some cases the rods were cut into short lengths separated by brass.

Mr. Boys asked how the nickel "stress" instrument compared in clearness and loudness with an ordinary telephone.

Mr. Shelford Bidwell had tried a nickel telephone with a mica diaphragm ; depending not upon mechanical stress, but magnetic strain. It did not work well.

Dr. Chree thought the "stress" telephone might possibly be improved by choosing the right strength of magnetic field.

Mr. Appleyard said the arrangement was interesting historically, because it was, mechanically, almost identical with the original instrument used by Philip Reis as a "receiver." The authors had succeeded in getting it to work as a "transmitter." Their success was probably due to the rapidity with which the magnetization of nickel responded to very small changes of stress or current. The Post-Office electricians had tried to introduce nickel cores into relays, on account of its magnetic sensitiveness; the results, he believed, had not been very satisfactory.

Mr. T. A. Garrett, in replying, said the "stress" telephone gave better articulation than an ordinary "watch" telephone, but the sounds were feebler. There seemed to be a field-strength proper to the instrument ; he had noticed that the articulation was clearer with three cells than with six.

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XI. *The Formation of Mercury Films by an Electrical Process.* *By* Rollo Appleyard[*].

If a sheet of gelatine, damp leather, or similar permeable substance is used as a separating diaphragm between two bodies of mercury, and a current is sent through it, a film of mercury is deposited upon the surface connected to the positive pole of the battery. And the film remains on the diaphragm after removal from the apparatus.

If the diaphragm is replaced in the apparatus and subjected to a current in the reverse direction, the film vanishes from that surface, and a second film appears on the other side. That is to say, the film is always on the side of the diaphragm connected to the positive pole of the battery, and there is no film on the negative surface.

In this way I have deposited mercury-films upon such different substances as filter-paper, plaster-of-paris, Woodbury-type-gelatine, porous earthenware, asbestos-paper, and sheep's skin. As a rule, the only preparation of the substances is damping with distilled water. Asbestos mill-board is one of the best materials to operate upon.

The coating of mercury formed by this electrical process is far more perfect than any I have been able to obtain by the methods of "pickling" or contact-pressure. I have no definite views as to the cause of the phenomenon, but I am inclined to regard it as a secondary action, the result of electrolysis, aided to some degree by electric osmosis. The experiments of Mr. C. K. Falkenstein[†] upon the electric tanning of leather are evidence in favour of the osmosis theory. I understand also that electric osmosis has lately been used for administering cocaine, replacing the ordinary hypodermic injections.

It will be remembered that in some experiments[‡] I recently described to the Physical Society a globule of mercury, electrified in a partially conducting liquid, sent forth a tentacle in a direction towards the positive electrode. It is perhaps significant that in the present experiments the film is always on the positive surface. In the case of liquid "coherers" it is possible to conceive that, under the influence of steady or transitory currents, mobile conductors attempt to creep in this fashion through the dielectric films. And, again, though the effect is probably very slight and entirely masked by electrolytic disturbances, sea-water may thus penetrate the dielectric at a "fault" in a cable. The film-forming process may be of practical utility where large surfaces are required to be coated with mercury, as in gold extraction; and possibly in the preparation of plates for primary or secondary batteries.

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[*] Read May 14, 1897.

[†] Journ. Inst. Elec. Engineers, vol. xxii. p. 225 (1893).

[‡] "Liquid Coherers and Mobile Conductors," *ante*, p. 72.

The following experiments are given in illustration of the action :—

(1) A sheet of filter-paper, damped with distilled water, is folded upon itself four or five times. It is then laid flat upon a conducting surface. An indiarubber ring is pressed down on the folded paper, and mercury is poured in to fill the ring. A battery of 200 volts is then connected for about three minutes between the mercury and the lower conducting plate, the mercury being positive. After stopping the current the ring is removed from the filter-paper and superfluous mercury is poured off, leaving a circular film upon the upper surface of the paper. On unfolding the sheet, a blackish substance is found distributed through it, graduated from the positive towards the negative pole.

(2) If a sheet of tinfoil is included within the folds of filter-paper, while the battery is applied, the tinfoil becomes perforated with pin-holes. This, however, must not be taken *per se* as a proof of any direct transition of metallic mercury through the diaphragm; for if, instead of the mercury contained in the ring, a sheet of metal is used as the upper electrode, pinholes still appear in a sheet of tinfoil included in the folds of damp filter-paper. Moreover, if a sheet of tinfoil is used as the upper electrode, this also becomes perforated.

(3) Again, if the sheets of tinfoil are removed, and a gold coin is used for the upper (positive) electrode, upon the top of the damp filter-paper, the current produces a gold discoloration which penetrates the folds. I have not yet had time

to examine this result, but it seems to throw some light upon the formation of metallic lodes and veins. And although the discoloration is the effect of a steady current, and not sparks it may help to explain the "inductoscripts"* of Mr. F. J. Smith.

DISCUSSION.

Dr. S. P. THOMPSON said he did not know of any other examples of an *anode* being more active—mechanically—than the *kathode*, except the electric arc. He was surprised that the film should appear on the positive surface.

Mr. SHELFORD BIDWELL thought selenium presented in some of its actions an example of the *anode* being thus active.

Prof. AYRTON said that if a vessel containing a substratum of mercury amalgam was filled up with water in which gold crushings were washed, the gold descended into the amalgam. This, however, might be partly due to gravity, and partly to simple electrolysis.

Mr. APPLEYARD said he had no definite views as to the formation of the films. He believed it to be a secondary effect of electrolysis, aided by electric osmosis. The experiments of Mr. C. K. Falkenstein upon the electric tanning of leather, and the early results of M. Perret, helped the idea of electric osmosis; they were not sufficient, however, to justify that theory without further research. A careful chemical analysis of the deposits left in the folds of filter-paper would be the best guide.

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* "Inductoscripts," by F. J. Smith, Proc. Phys. Soc. vol. xi. p. 353 (1892).

XII. *An Instrument for the Comparison of Thermometers.*

By W. WATSON, *B.Sc., A.R.C.Sc., Assistant Professor of Physics at the Royal College of Science, London*[*].

IN most investigations where it is necessary to measure temperature it is convenient to use a small thermometer, often of short range, particularly suited for the purpose. The only practicable method of determining the errors of such a thermometer is by comparing its readings with those of a standard thermometer. This comparison is in general rendered very difficult on account of the two thermometers differing both as to the length of the scale and the capacity of the bulb. Hence if, as is essential, the bulbs are placed close together in a water-bath, a considerable length of one of the mercury columns must be out of the bath, involving an uncertain correction for cool column. Again, the difference in size of the bulbs renders the lag of the thermometers different, so that unless the temperature of the bath is kept absolutely constant for some time we cannot be sure that *both* thermometers have reached a steady temperature. These objections make the use of a water-bath at temperatures over about 20° C. both troublesome and unsatisfactory, and this paper is written in order to describe an instrument which the author has designed for comparing some thermometers required for some other work, and which has worked so satisfactorily that it may be of some use to others.

The principle employed is that of using a vapour-jacket in order to obtain different constant temperatures, as recommended by Ramsay and Young (Journ. Chem. Soc. xlvii. p. 640, 1885). A general view of the instrument is shown in fig. 1. A glass tube AB, closed at the top and open below, about 76 cm. long and 2·5 cm. in diameter, passes up inside a second glass tube of about the same length and 4·7 cm. in diameter. The space between these two tubes is at the bottom closed by an indiarubber cork C (figs. 1 and 2), while at the top the inner tube is kept in place by three small pieces of glass rod fused on and forming a triradiate star. The upper end of the outside tube is fused on to a Liebig condenser D, the upper end of the condenser being connected by a rubber joint fitted with a glycerine seal to a manometer E, a stoppered funnel F, and a large glass bottle which acts as a reservoir. This bottle is packed round with cotton-wool and is enclosed in a wooden box G. The air can be exhausted from or admitted to this bottle by means of a three-way tap H. Some mercury is placed on the top of the cork C (fig. 2) to prevent the liquid used to form the vapour touching the cork. The liquid is heated by means of a spiral of fine uncovered platinum wire, the terminals being fused into glass tubes which pass down through the mercury and the cork.

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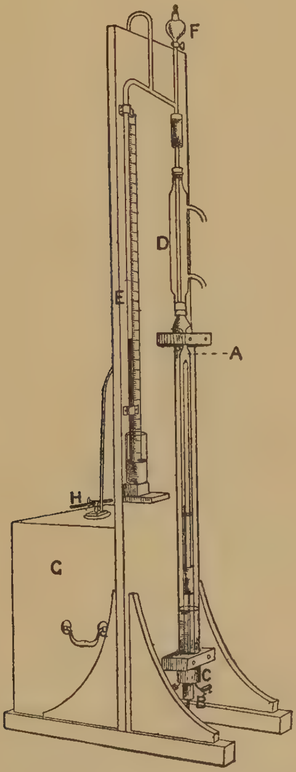
[*] Read May 14, 1897.

Fig. 1.



Attempts to boil the liquid in an auxiliary bulb were rendered futile on account of the excessive bumping which took place after the liquid had been boiling for a day or two. The heating by a platinum wire, however, entirely obviates all bumping, and might be of use in determining the boiling-point or in distilling liquids which are inclined to bump, so long as they are not electrolytes.

Fig. 2.

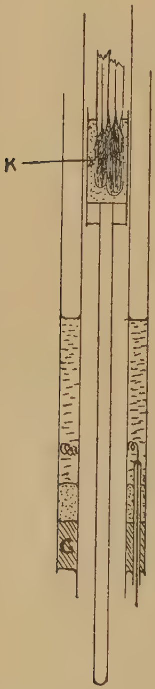
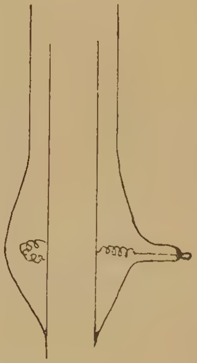


Fig. 3.



The vapour of the boiling liquid rises between the inside and outside tube, is condensed in the condenser D, and runs down to the bottom again.

The thermometers are fastened together, their bulbs resting in a small glass vessel K (fig. 2) which is filled with mercury. This vessel is supported by a glass rod which rests on the table on which the instrument stands. To remove the thermometers the instrument is drawn to the edge of the table till this rod can be lowered and with it the thermometers. The scales are read by means of two telescopes, it being quite easy to see through the vapour if suitable liquids are used. The three liquids recommended by Ramsay and Young, viz. carbon bisulphide (20° to 46°), ethyl alcohol (46° to 79°), and chlorobenzene (79° to 120°) do very well. Water cannot be used as it forms drops on the inside of the glass so that the thermometers cannot be read. For temperatures higher than 120° the indiarubber cork closing the tube will not do, but the arrangement shown in fig. 3 may be used. The disadvantage of this form is that it can hardly be made by any but a professional glass-blower; the first form, on the other hand, can easily be put together by anyone who can do the most rudimentary glass-blowing.

In order that the indiarubber cork may form a thoroughly air-tight joint, it must be well cleaned with benzene, coated with indiarubber solution, and put in place while the solution is wet. When the solution has got thoroughly dry, which takes four or five days, the joint will be quite air-tight.

With such an arrangement the thermometers can easily be maintained at a temperature constant to within 0°·01 for three or four hours together. The manometer is only used to adjust the temperature to the desired point, a table of vapour-pressures being employed. In about half an hour after altering the pressure, and hence the temperature, the thermometer readings become quite constant. The instrument once started can be left entirely to itself; the only thing the observer has to do is to read the thermometers, then let a little more air in to get the next higher temperature. He may then go away for half an hour, and when he returns will find the temperature constant and can take the new readings.

DISCUSSION.

Prof. AYRTON thought the apparatus would come into extensive use; it did away with errors arising from differences of length of thermometer stems, it left no question as to the equality of temperature of the two bulbs, and there was no probability of error due to a difference of thermal "lag" in any two thermometers.

Mr. Watson, in replying to a question of Prof. Perry's, said the fact of using indiarubber joints limited the available range of temperature. Working with blown joints, Ramsay and Young had found no difficulty in their vapour-density experiments at higher temperatures.

XIII. *On the Isothermals of Isopentane.*

By J. Rose-Innes, *M.A.*, *B.Sc.*[*]

[Plate IV.]

The recent publication by Prof. Sydney Young of a long series of researches on isopentane (Proc. Phys. Soc. Session 1894–95, pp. 602–657) offers a great opportunity to those interested in the theory of gases. Since isopentane is a saturated hydrocarbon, there seems to be a reasonable hope that we are here dealing with a substance which will not tend to form complex molecules at low volumes, and whose behaviour may therefore be treated as normal; and this fact, together with the wide range of volume over which the experiments have been conducted, renders Prof. Young's results well fitted to test the various formulæ that have been from time to time proposed. Among these formulæ there is none more important than that formerly suggested by Prof. Young himself, in conjunction with Prof. Ramsay, that the pressure of a gas kept at constant volume is a linear function of the temperature; and the bearing of the experimental results with isopentane upon this formula is fully considered by Prof. Young in his paper. Accepting the formula, and writing it as

$$p=b\mathrm{T}-a,$$

where b and a are functions of the volume only, the values of b and a for a large number of volumes are given in the paper (*loc. cit.* pp. 650–655), and they are sufficiently numerous to enable us to fully test any algebraic expression that endeavours to represent them. I spent a considerable amount of time examining the values of a and b, testing the formulæ that have been proposed by various physicists, as well as others of my own devising, without arriving at any that gave complete satisfaction; and it occurred to me afterwards that possibly more definite results could be secured by examining some physical quantity which depended upon both a and b than by examining b and a separately by themselves. Thus if we accept Ramsay and Young's linear law, there will be one and only one temperature for each volume at which the gas has its pressure equal to that given by the laws of a perfect gas. In effect, if we put

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[*] Read May 28, 1897.

$$p=b\mathrm{T}-a$$

as giving the actual pressure, we may also write

$$p=\frac{\mathrm{RT}}{v}+\left(b-\frac{\mathrm{R}}{v}\right)\mathrm{T}-a,$$

and this shows that $p=\frac{\mathrm{RT}}{v}$ if we take $\mathrm{T}=\frac{a}{b-\mathrm{R}/v}$. Call the temperature so found τ; the values of τ have been calculated and are given in the following table (p. 128); the results above vol. 400 are not included, as they vary within such wide limits owing to experimental errors.

In calculating this table, the value of R was taken $=\frac{1}{\cdot 001158}$.

An examination of the table shows that the temperature in question is very much the same for all large volumes down to about vol. 8. Of course the actual numbers obtained vary a good deal, but these variations are without method, sometimes in one direction and sometimes in another, and when the numbers are plotted against $v^{-\frac{1}{3}}$ it appears to me impossible to tell from an inspection of the diagram whether the value above vol. 8 is on the whole increasing or decreasing. These variations may therefore be attributed to experimental error; and they may be to a large extent got rid of by employing "smoothed" values of b and a, as was done in Ramsay and Young's paper on ether (Phil. Mag. xxiii. p. 441). Prof. Young has not given any smoothed values of b for isopentane, however, and I have thought it better not to use my own smoothed values, so as to employ numbers that are above the suspicion of bias towards any particular theory.

Volume.	τ.	Volume.	τ.	Volume.	τ.
350	851·9	22	835·8	4·3	804·6
300	835·4	20	845·9	4·0	799·1
260	820·8	19	854·7	3·8	792·3
230	825·6	18	848·7	3·6	777·3
200	969·6	17	850·9	3·4	764·0
180	935·6	16	845·3	3·2	737·2
160	846·7	15	854·1	3·0	715·5
140	787·0	14	848·9	2·9	696·5
120	808·9	13	847·8	2·8	675·9
100	804·1	12	844·1	2·7	656·4
90	793·8	11	840·1	2·6	637·7
80	773·4	10	843·5	2·5	615·2
70	854·7	9·5	846·7	2·4	593·7
60	876·2	9	838·3	2·3	565·9
50	871·6	8·5	834·5	2·2	541·1
45	853·1	8	833·7	2·1	511·2
40	816·2	7·5	833·2	2·0	478·7
36	832·8	7	832·2	1·9	446·3
33	820·0	6·5	828·1	1·8	408·6
30	833·7	6	827·2	1·7	367·2
28	835·1	5·5	819·8	1·6	318·4
26	846·7	5	813·7		
24	848·0	4·6	809·5		

The mean value of τ for all volumes above 8 is 842·4; and this number could be taken as the true value for all such volumes without introducing any serious error. Below vol. 8, however, the value of τ distinctly diminishes with the volume; and by the time the critical point is reached, τ is not more than about 804. It is not easy to find a formula for τ which will give the experimental results throughout the whole range of volume; but whatever may turn out to be its proper algebraic expression, there appears to be little doubt that its initial decrease from the maximum value varies as a higher power of the density than the first.

The result appears to me to be of importance, since it is by itself sufficient to reveal the inadequacy for isopentane of many of the gas-formulæ that have been proposed. We have seen that

$$\tau = \frac{av}{bv - R}.$$

Suppose that we can put

$$b=\frac{R}{v}+\frac{B_2}{v^2}+\frac{B_3}{v^3}+\text{ higher powers of }\frac{1}{v},$$

$$a=\frac{A_2}{v^2}+\frac{A_3}{v^3}+\text{ higher powers of }\frac{1}{v};$$

Then

$$\tau=\frac{A_2+A_3v^{-1}+\ldots}{B_2+B_3v^{-1}+\ldots},$$

$$=\frac{A_2}{B_2}+\frac{A_3B_2-A_2B_3}{B_2^{\,2}}v^{-1}+\text{higher powers of }\frac{1}{v}.$$

If b and a in any gas-formula can be expanded in powers of v^{-1} as assumed, and if $A_3B_2-A_2B_3$ is different from zero, which will in general happen, the decrease of τ from the maximum value will be as v^{-1}; so that the proposed formula is inadmissible.

Let us now return to the quantities a and b. The values of $\frac{1}{av^2}$ were plotted against $v^{-\frac{1}{3}}$ by Prof. Young, and the resulting diagram is given in his paper (*loc. cit.* p. 653). The curve determined by the points plotted would seem to be of a somewhat complicated character, and I do not think it possible to obtain any simple formula that will reproduce it entirely. On the other hand, there seems to be considerable evidence of discontinuity in the neighbourhood of vol. 3·4; and even if there is not discontinuity in the true mathematical sense of the word, there appears to be such a rapid alteration of behaviour as to amount in practice to the same thing. The easiest plan is to treat the curve as consisting of two parts, the formula passing abruptly from one expression to another somewhere in the neighbourhood of vol. 3·4. Whether this accurately represents what takes place in nature is uncertain, but there is no doubt that it immensely simplifies the problem. We may therefore confine our attention to volumes above 3·4; and of the formulæ already proposed I found the best to be that suggested by Mr. W. Sutherland, who has given a gas-formula equivalent to putting

$$a=\frac{l}{v(v+k)},$$

where l and k are two constants characteristic of the gas

(Phil. Mag. xxxv. p. 215). I have taken the following values of the constants

$$l=5{,}420{,}800, \qquad k=3\cdot 636.$$

The next step was to find a formula for b. Now

$$b=\frac{\mathrm{R}}{v}+\frac{a}{\tau},$$

and the conditions already discovered with respect to the algebraic expression for τ considerably restrict the field of research; this is an advantage, as it lessens the amount of arithmetical work to be performed. I found that fairly good results could be obtained by putting

$$b=\frac{\mathrm{R}}{v}\left(1+\frac{e}{v+k-\frac{g}{v^2}}\right),$$

where R and k have the values already given and

$$e=7\cdot 473, \qquad g=6\cdot 2318.$$

By combining the formulæ given above for a and for b, we obtain as the formula for the isothermals

$$p=\frac{\mathrm{RT}}{v}\left(1+\frac{e}{v+k-\frac{g}{v^2}}\right)-\frac{l}{v+k},$$

where R, e, k, g, and l are constants, and have the values already given.

In order to test this formula it is desirable to draw a system of isothermals, but if this be done in the ordinary Andrews' diagram the result is not satisfactory, as the range of p is so large. It was found possible to obtain a good diagram, however, by calculating pv and plotting it against $v^{-\frac{1}{3}}$; the calculated isothermals are shown as continuous lines, while the experimental values are put in as dots. It will be seen that there is a fair agreement between calculation and experiment down to about vol. 3·4. Below this volume there is no longer any agreement; we should naturally expect such a result, since the formula for a admittedly holds up to the neighbourhood of vol. 3·4 only. Therefore, even if the calculated isothermal were to agree with the found isothermal below vol. 3·4 for some one temperature, this would only happen by a compensation of errors, and could not occur for any second temperature.

It will be noticed that in the neighbourhood of vol. 16 there is a sensible divergence between the calculated and found isothermals amounting to slightly over 1 per cent. This divergence is certainly unsatisfactory as far as it goes; but differences just as great have occurred in the past in inquiries of this kind between the results of independent observers experimenting on the same substance. For the present, then, and pending the confirmation of Prof. Young's results by some other observer, we may take the simple formula given above as representing all that we certainly know concerning the behaviour of isopentane under the conditions of volume specified.

The formula proposed may be employed to calculate the critical constants; this may be done by a method depending solely on Ramsay and Young's linear law. Let us take the equation

$$p=b\mathrm{T}-a,$$

and differentiate it with respect to v, keeping T constant,

$$\frac{dp}{dv}=\mathrm{T}\frac{db}{dv}-\frac{da}{dv}.$$

Differentiate again

$$\frac{d^2p}{dv^2}=\mathrm{T}\frac{d^2b}{dv^2}-\frac{d^2a}{dv^2}.$$

At the critical point $\frac{dp}{dv}$ and $\frac{d^2p}{dv^2}$ vanish together, so we have

$$\mathrm{T}\frac{db}{dv}-\frac{da}{dv}=0,$$

$$\mathrm{T}\frac{d^2b}{dv^2}-\frac{d^2a}{dv^2}=0.$$

Eliminating T this gives

$$\frac{db}{dv}\frac{d^2a}{dv^2}-\frac{d^2b}{dv^2}\frac{da}{dv}=0.$$

When v has been calculated from this equation we may obtain T by putting

$$T = \frac{da}{dv} \div \frac{db}{dv},$$

and when v and T are known the original isothermal equation will give p.

In this way we may obtain the following results :—

$$\begin{cases} \text{critical volume} = 4{\cdot}5, \\ \text{critical temperature} = 191^\circ{\cdot}7 \text{ C.,} \\ \text{critical pressure} = 26250, \end{cases}$$

which agree fairly well with the numbers found experimentally by Prof. Young.

There is one of the isothermals in the set determined by our formula which is of peculiar interest. The biggest value of τ occurs when v is indefinitely large, its value being then $\frac{l}{Re}$. Let us call this quantity τ_0 for shortness, and let us see what form the isothermal for this particular temperature takes. We shall easily obtain

$$p = \frac{R\tau_0}{v} + \frac{lg}{v^3\left(v+k-\frac{g}{v^2}\right)(v+k)},$$

so that the pressure for a long range of volume is practically given by Boyle's law ; and when an increase does occur, it varies as the *fifth* power of the density. In the case of isopentane this is merely a deduction from theory, since the temperature required is 567° C. ; much too high to allow of any experiments being carried out for verification. Indeed, for most other substances, the temperature of minimum deviation from Boyle's law, being much higher than the critical temperature, is much too high to allow of any experiments being performed there. An exception may be made, however, in the case of nitrogen, which has a very low critical temperature. The following are the values of pv at 16° C. for nitrogen under different pressures determined by M. Amagat (" Mémoires sur l'élasticité et la dilatabilité des Fluides jusq'aux très-hautes Pressions," *Ann. Chim. Phys.* 6th series, vol. xxix. p. 107).

Pressure in metres of mercury.	*pv.*	Pressure in metres of mercury.	*pv.*
·76	1·0000	45	·9895
20	·9930	50	·9897
25	·9919	55	·9902
30	·9908	60	·9908
35	·9899	65	·9913
40	·9896		

From this table it is evident that at 16° C. for nitrogen we are only a short way off the temperature of minimum deviation from Boyle's law. At temperatures higher than the temperature of minimum deviation the gas would have *pv* increasing with pressure from the first, and this is what happens with hydrogen at ordinary temperatures.

DISCUSSION.

Prof. YOUNG said the diagrams representing the observed and calculated isothermals were probably the best ever obtained. Divergence among the values of τ was explained in part by the smallness of the angle between the theoretical isochor for a perfect gas, and the real isochor. The point of coincidence was difficult to define. Moreover, the values of τ were obtained from "unsmoothed" values of v. The evidence against the linear law consisted in a certain similarity in the shape of the different curves. It was not easy to see where experimental errors could come in. The peak was a very striking feature of the curves, and the agreement between the results with ether and those of isopentane was very remarkable. These two substances had their boiling-points close together, their critical temperatures close together, and their molecular weights nearly alike. The two substances not only agreed in each giving a peaked curve, but the peak corresponded to almost identical volumes. Prof Young hoped at some future time to examine normal pentane, and to determine whether τ was a constant for this substance also.

XIV. *On the Magnetic Properties and Electrical Resistance of Iron as dependent upon Temperature.* *By* David K. Morris, *Ph.D., formerly* 1851 *Exhibition Scholar, University College, London*[*].

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[Plates V. & VI.]

The changes brought about by heating in the various physical properties of iron have been, since the first investigations into the nature of the phenomenon of recalescence by Gore and by Barrett, the subject of much experimental work, which, during the last ten years, has been largely upon the relation of the magnetic properties of iron to temperature.

As long ago as 1879, however, Baur of Zürich [†] described experiments on an iron bar inserted while hot into a long magnetizing helix, which served to determine its permeability at any instant, the temperature being estimated from a knowledge of the law of cooling. This method, although rough, enabled him to show that in weak magnetic fields the permeability of iron rises, and that in strong ones it falls, with rise of temperature; and that as the critical point is reached the magnetic qualities of iron in fields of all strengths very rapidly disappear.

These results have been confirmed and extended by the researches of Ledeboer in 1888 [‡], and by those of Dr. Hopkinson, communicated to the Royal Society in 1889 [§]. In the latter, a ring-magnet was used, furnished with primary and secondary coils insulated with asbestos, and capable of withstanding a red heat, which temperature was obtained by means of a gas-furnace. Measurements could thus be made of the permeability of the iron core by the ballistic method, and the temperature was deduced from the resistance of the copper secondary winding.

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[*] Read May 14, 1897. Inaugural Dissertation for the Degree of Ph.D. at the University of Zürich.

[†] C. Baur, "Neue Untersuchungen über den Magnetismus," Wied. *Ann.* 1880, p. 394.

[‡] Ledeboer, *Journal de Physique* (2) vol. vii. p. 199.

[§] Hopkinson, "Magnetic and other Physical Properties of Iron at a High Temperature," Phil. Trans. Roy. Soc. 1889, A. p. 443.

M. Curie, in a more recent investigation [*], used a method by which the permeability could be obtained in intense magnetic fields. He experimented on a sample of iron in the interior of an electrically-heated porcelain furnace, and deduced its magnetization from the forces acting on it when placed in a non-uniform magnetic field. A thermo-electric couple served to measure the temperature.

The variations of magnetic hysteresis with temperature have been studied by W. Kunz of Darmstadt [†].

The relation of the electrical resistance of iron to temperature has also received the attention of several experimenters; curves expressing this relation, and extending to beyond the critical temperature, having been given by Kohlrausch in 1887 [‡], by Hopkinson in 1889 [§], and by Le Chatelier in 1891 [‖].

General Description of the Method of Experiments.

By the method used in the present work, measurements of the electrical resistance and of the magnetization of the iron could be made simultaneously.

It consisted in the employment, as in Dr. Hopkinson's experiments, of a ring-magnet, whose insulation was calculated to withstand a white heat. The core of this ring-magnet was formed of a length of *insulated* iron wire or strip, whose ends projected out of the ring, so that the resistance of the iron in the core could be measured; and *in this core* was imbedded an insulated platinum wire, from the resistance of which the temperature of the iron could be deduced.

The heating was carried out electrically. For this purpose the ring (which was made quite small, about an inch in diameter) was covered with a non-inductively wound layer of insulated platinum wire. By passing a suitable current through this wire the temperature could be raised to any required extent. The heat was thus generated exactly where wanted; and by wrapping the ring thickly with asbestos the loss of heat by radiation was made small, and the temperature in the interior fairly uniform.

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[*] P. Curie, "Propriétés magnétiques des corps à diverses températures," *Ann. de Chim. et de Phys.* July 1895, p. 289.

[†] W. Kunz, *Elektrotechnische Zeitschrift*, 1894, p. 194.

[‡] W. Kohlrausch, Wied. *Ann.* vol. xxxiii. p. 42.

[§] Hopkinson, Proc. Roy. Soc. vol. xlv. p. 457.

[‖] Le Chatelier, *Comptes Rendus*, vol. cx. p. 283.

In the experiments of M. Curie, 1500 watts were absorbed in obtaining a temperature of 1350°: by the above method, 73 watts (a current of 4·6 amperes with 16 volts across the terminals) was found sufficient to maintain the ring at 1150°; and 45 watts sufficient at 800°, *i. e.*, above the critical temperature of the iron. Dealing thus with comparatively small currents, the accurate regulation of the temperature was rendered simple; whilst absolute constancy for long periods of the heating current, and therefore of the temperature, was, with a set of accumulators in good order, not difficult to attain.

To avoid oxidation of the iron, the ring-magnet was placed in a glass vessel, and the electrical connexions brought through a well-fitting cork rendered air-tight by a thick layer of sealing-wax poured over it while hot. The oxygen contained in this closed space could, previous to the experiments, be absorbed by an auxiliary coil of bare iron wire, heated to bright redness by an electric current. In the later experiments the jar was, as a preliminary, exhausted by a small air-pump to about a third of an atmosphere; during the heating, the pressure might rise above that of the atmosphere, in which case the excess of heated air was allowed to escape from a glass tube dipping into mercury; and this at other times served as a gauge to show whether the glass vessel was really air-tight.

Details of the various Ring-Magnets.

During the course of the experiments, four ring-magnets were made.

The first came quickly to grief through accidental overheating and partial oxidation of the iron. The oxide seems to have combined with the silicates of the insulating materials, forming a kind of slag, for the ring, on taking to pieces was simply a collection of platinum wires buried in a brown glassy substance with a little iron left in the core.

The experiments with the second ring (whose core was formed of a length of ordinary soft iron wire) were more successful; but the results, though interesting, have not been thought worth including in this paper. The experience gained in the construction and use of the first two ring-magnets was turned to account in the two later ones.

Platinum wires were exclusively employed for the windings of the ring-magnets, as no substance either magnetic or fusible below 1200° was admissible. These wires, after use in one ring, were used again, with fresh insulation, in the next.

Iron cores.—The cores of the third and fourth ring-magnets (referred to in this paper as specimens A and B respectively), were formed from strips, one chosen from each of two groups of iron samples kindly procured for the author by Mr. R. Jenkins from Messrs. Jos. Sankey and Sons, of Bilston. The sample strips were 1 cm. in width; they were described as follows:—

> Specimen A.—"Charcoal Iron, No. 4 Quality; thickness ·02 in."
>
> " B.—"Best Transformer Quality ◇; thickness ·014 in." *

These specimens proved, after careful annealing, to be exceptionally good both as regards high permeability and low hysteresis.

Platinum Thermometer Wire. Method of deducing Temperatures.—The specimen of pure annealed platinum wire used in the measurement of temperature was from the firm of W. C. Heräus, of Hanau, near Frankfurt-a.-M. The same piece of wire was used in both the later ring-magnets. Its diameter was ·0453 cm., its specific resistance at 0° was 10,110 C.G.S. units, and its temperature-coefficient, which was remarkably high, was $\alpha_{0^\circ-100^\circ} = \cdot 00386$.

The resistance-temperature curve was taken by mounting the platinum wire (carefully re-annealed by heating to bright redness in a mass of asbestos) on a small cylindrical mica frame, and measuring its resistance at intervals of about 30°

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* This iron, according to the above firm, gives on analysis 99·925 per cent. of iron by difference, the impurities (·075 per cent.) being distributed between carbon, phosphorus, and silicon, with a trace of manganese. It is as pure Swedish iron as can be produced commercially

up to 200° C. in a bath of linseed oil. The temperature was measured by a standard thermometer which was carefully checked at 0° and 100°. The oil was vigorously stirred, and observations of resistance were only made when the temperature was very nearly constant.

The experiments of Callendar and Griffiths have established the fact that, for pure annealed platinum wire, if $\frac{1}{100}$th part of the increase of resistance which it experiences in passing from 0° to 100° be called the increment for one "platinum degree," so that the resistance of a platinum wire is a linear function of its "platinum temperature," pt°; then, if t° is the actual temperature on the air-thermometer scale, the difference $t^\circ - pt^\circ$ can be expressed over wide ranges of temperature by the form $a + bt + ct^2$, which in this case must be reducible to the form $t^\circ - pt^\circ = \delta\{(t/100)^2 - t/100\}$.

It was thought that, in the present investigation, an error in the determination of the absolute values of the higher temperatures of even several degrees was of comparatively small moment where the recalescence point for different specimens of iron is known to differ by 50° or more. The resistance-temperature curve for the platinum thermometer wire was not determined directly above 200°, but was extrapolated by the help of Callendar's formula. The value of δ for this wire was 2·10.

The *absolute* values of the temperatures given must therefore be accepted with caution. They may be wrong by as much as 5° in the neighbourhood of 800°, the error rising with the square of the temperature. For the present purpose, however, it is the *relative* accuracy which is of importance, and this is of a much higher order.

To obtain reliable measurements of temperature, it was necessary that the thermometer-wire should satisfy the following conditions:—(1) Its temperature must not differ sensibly from that of the iron which it is intended to measure; (2) its constants must not alter in any way during the heating of the ring; (3) its resistance must be measured with sufficient precision.

(1) The first condition was the most difficult to satisfy: and it was only by actually burying the thermometric wire in the core itself, and by having the temperature very nearly

constant before taking any observations, that the temperature of the thermometer-wire and that of the iron core were brought to a satisfactory coincidence.

(2) With regard to the second condition: the resistance-temperature curve of the thermometer-wire, besides being taken previous to experiments, was also taken after those with each of the specimens A and B; in neither case could any alteration either in the resistance-coefficient or in the value of δ be detected. (Alterations of these constants which had taken place with former thermometer-wires were thought to be due to the fact that the insulation in contact with them had not been previously decarbonized (see p. 142).)

One alteration, however, did take place, particularly during the last annealing of specimen B, when a temperature of 1150° was reached and maintained for some time: the resistance of the thermometer-wire rose (on this occasion as much as 1 per cent.); though, as stated above, the form of its resistance curve remained throughout unchanged.

Messrs. Heycock & Neville * have repeatedly noticed such changes, though slight in amount, in their resistance pyrometers; and conclude that they were due to strains set up by the permanent expansion of the mica frames on which their wires were supported. It is possible that such increase of resistance may also be due, as Prof. Weber has suggested, to evaporation of the platinum at these high temperatures; none of the author's platinum wires retained, after *prolonged* heating, their bright metallic surface †.

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* Heycock and Neville, "On the Determination of High Temperatures by means of Platinum Resistance Pyrometers," Trans. Chem. Soc. 1895, p. 160.

† Since the above was written, Mr. E. H. Griffiths, F.R.S., has kindly drawn my attention to the fact that his own platinum thermometer wires on mica frames, when heated for considerable periods at temperatures of 1200°–1400°, do not thus lose their former bright surface. It is therefore rather improbable that in my case the dull surface should point to any serious evaporation of the platinum. I have in this connexion one or two observations which I had not given as I was in doubt as to their reliability:—The mean *sectional area* of the platinum thermometer wire, as measured by the specific gravity method, diminished from ·001182 cm.^{2} to ·001162 cm.^{2} during use in Rings A *and* B; the *length* increased from 31·33 to 31·56 cm. during the heating in Ring B, and during this same heating the *weight* of the wire did not alter.—D. K. M.

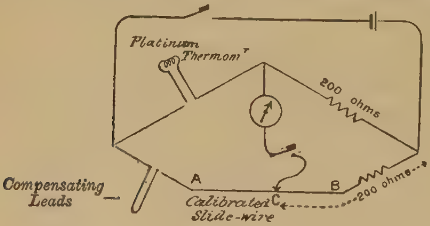
In reducing the results, it has been assumed that this alteration of resistance, which was generally very small, took place entirely during the time that the ring was at its highest temperature.

(3) In order to satisfy the third condition, it was not only necessary that precise measurements of resistance should be possible ($\frac{1}{20}$ per cent. represents more than $\frac{1}{2}$° at 1000°), but also that the resistance of the thermometer-wire and that of its compensating leads should be measured very nearly simultaneously.

To obtain the nett resistance from a *single observation*, the following arrangement (fig. 1) was employed, which is similar to, though not identical with, that used in Callendar's platinum thermometers.

The platinum thermometer-wire and leads formed one arm of the Wheatstone's bridge, and the compensating leads (made exactly equal to those of the thermometer), plus a portion AC of the slide-wire, formed a second arm equal to the first. The other two arms contained equal resistances; the 200-ohm coil of a resistance-box in the one; and in the other the smaller coils of the box (together making 200 ohms) with just so many tenths of an ohm short-circuited as were in the remainder CB of the slide-wire. It is clear then that, if C is the position of balance, the resistance AC is equal to the nett resistance of the thermometer-wire.

Fig. 1.—Form of Bridge for Temperature Measurements.



The resistance of the slide-wire, whose calibration curve was very uniform, was about 1·4 ohms. This was approximately that of the thermometer-wire at 1200°, so that the balancing point for all lower temperatures was within the range of the metre-long slide-wire. The resistance per unit length of the latter was conveniently obtained by inserting a standard ohm in the place of the thermometer-wire.

The above method was also applied to the measurement of the resistance of the iron in the core.

Insulation.—The chief difficulty of the method of these experiments lies in so insulating the wires themselves and the various layers of the ring-magnet from one another, that the insulation shall still remain good at the highest temperatures reached.

The only material which is practically applicable to the insulation of the wires is *asbestos paper*; this paper, which though fairly thin is by no means uniform, was used in the form of long narrow strips wound spirally over the wires to be insulated. After considerable practice, it was found possible to cover wires even as small as ·3 mm. in diameter in this way with a reliable, and fairly thin and uniform covering; for such wires, a strip about 4 mm. wide was found to be most suitable.

Asbestos paper must of necessity contain starch or something similar to hold it together; and experiments proved that, when heated in a non-oxidizing atmosphere, the resulting carbon deposit reduced the insulating power of the asbestos to a very low value. This difficulty was overcome by carefully burning out all the hydrocarbons in one layer of insulation, leaving, so to speak, nothing but the asbestos 'ash,' a white and very brittle covering; and then cautiously over-covering this with a layer of ordinary asbestos paper.

In the first experiments, the leakage between the heating-coil (with 10 volts across its terminals) and the secondary winding was sufficient, when the ring was at 1000°, to drive the astatic needle of the sensitive galvanometer included in the secondary circuit almost round to its stops; and though the needle could be easily brought back artificially to its zero position again, the leakage was uncertain enough in amount to render ballistic observations at this temperature quite out of the question. The introduction of the above-described improvement in the asbestos insulation reduced this leakage to about $\frac{1}{100}$th part of its former amount; but it still represented a not very constant zero error of several centimetres.

With the object of still further reducing the leakage, *mica* was tried as insulation between the layers. This plan ultimately succeeded very well; but the difficulty of applying a reliable layer of mica insulation to a surface of "double curvature," such as that of a small ring, without the assistance of any adherent whatever, appeared at first insurmountable. Latterly, however, it was found that if the mica insulation for the "doubly-curved" parts was applied in the form of small suitably-shaped pieces, previously heated in a Bunsen flame * and then bent or continuously broken by pressing over a rounded edge, then, by putting each piece of mica into place as the coil of insulated wire above it was wound, a reliable layer of mica insulation could be obtained without much trouble.

Mica was also used in the later ring-magnets both for the insulation of the iron strip and for that of the thermometer-wire, in a way about to be described; it was not only put between the turns of the iron strip, but was also placed next to it, beneath the asbestos covering; since any carbon deposited from the latter would, if in contact with the glowing iron, combine with it and alter its character.

It is easily shown that no appreciable error in the measured resistance of the iron or thermometer-wire would arise from a conductivity of the insulation sufficient to give rise to even a large deflexion of the ballistic galvanometer needle.

Construction of Ring-Magnets.—The rings containing specimens A and B were almost identical in construction, differing mainly in the superiority of the insulation of the later ring. They were made in the following way (see figs. 2–4):—

A sample iron strip was chosen, and a narrow band 2 mm. wide was cut by a cold chisel out of the middle, leaving a specimen consisting of two parallel strips each 4 mm. in width joined at one end. The mean section of this double strip was determined by the specific gravity method. It was then insulated with mica and covered with asbestos as shown in fig. 2.

The insulated thermometer-wire was next laid in the space between the two iron strips, separated from them by slips of mica laid crosswise, and covered over by a plain mica strip (fig. 3). The whole was then rolled up on a temporary wooden form into a ring of four or five turns (fig. 4).

---

* This diminishes the tendency of the mica to split, and seems to do so by burning off the raw cut edges.

Thus, though asbestos was used in the core as insulation, it was the mica alone that was relied upon.

The ends of the iron specimen were then bent so as to leave the ring at right angles, and form a "stem" about 10 cms. in length. A portion of the same iron specimen resembling

Fig. 2.—Preliminary Stages. Half full size.

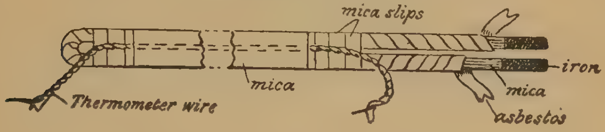


Fig. 3.—Section. Full size.

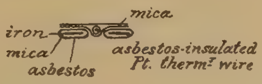
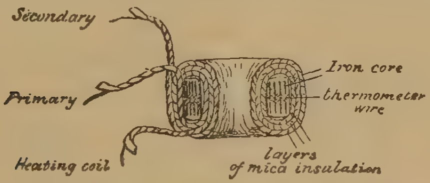


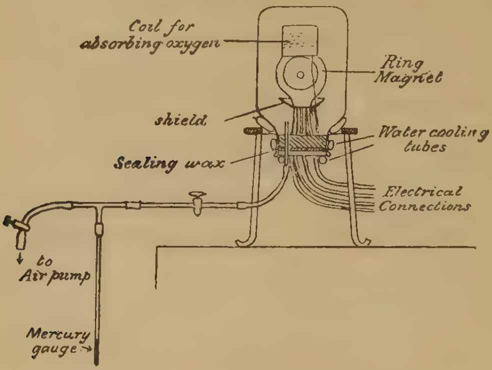
Fig. 4.—Section of completed Ring. Scale ½ full size.



this stem both in form and insulation, but only leading just up to the ring and back (see fig. 6), was bound on beside the stem that compensation for its resistance might afterwards be made. The ends of the platinum thermometer-wire, together with a similar compensating resistance formed of a piece of the same platinum wire, were also led up the stem, which was in all parts most carefully insulated with mica.

The next step was to overwind the ring with three layers of platinum wire to serve respectively as magnetizing coil or primary, as heating coil, and as secondary. These wires, each of which, like the thermometer-wire, was doubly-insulated with asbestos in the way previously described, were respectively 110, 150, and 100 cm. in length, and ·6, ·45, and ·3 mm. in diameter. Mica insulation separated the several windings from each other and from the core.

Fig. 5.—Showing Ring in Glass Vessel.



The heating wire, wound back on itself to avoid magnetizing effect, was passed once or twice round the lower part of the stem to prevent that local abstraction of heat to which the stem would otherwise have given rise.

The secondary winding was wound last of all; it being possible in this way to secure better insulation, particularly from the primary, from which even a small leakage into the secondary was much more serious than any leakage from the heating coil; for in the former case every ballistic throw due to a change in the primary current would be associated with a change of zero, while in the latter case no such change would occur.

The ring was afterwards successively covered with a layer of mica, a layer of asbestos, a covering of iron-wire gauze, and, finally, with a jacket consisting of three coverings of soft asbestos millboard. The iron gauze, which became red hot when the ring was being used at high temperatures, was placed there to intercept oxygen on its way in.

The *oxygen-absorbing coil* was, however, quite distinct from this. It consisted of two bare iron wires, connected in parallel, wound side by side on a cylindrical asbestos frame, and surrounded by a loosely-fitting asbestos mantle. Before the experiments they were heated by an electric current to a temperature at which they would readily absorb oxygen.

The coverings of the ring-magnet formed a mass of asbestos about 6 cm. in diameter ; a small hole through them enabled one to see what sort of temperature had been reached in the middle of the ring.

The coil for absorbing oxygen was then made fast to the ring-magnet, and the leading wires brought through a cork (previously boiled in linseed oil), which was then inserted into the glass jar and covered with melted sealing-wax (fig. 5). Finally, the terminal wires were soldered to those leading away to the rest of the apparatus.

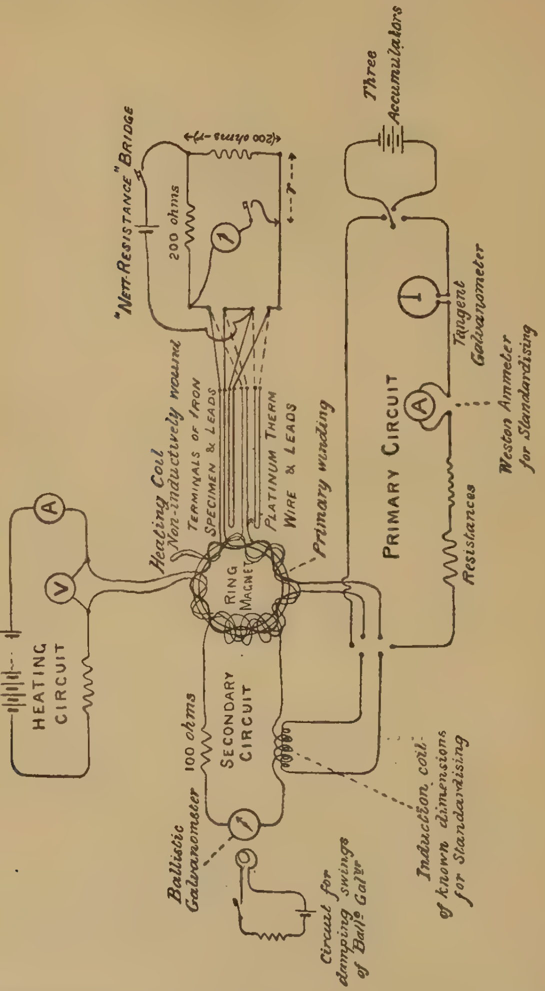
The following particulars relate to the last two ring-magnets constructed :—

Core of ring .... specimen	A.	B.
Description of iron	Charcoal Iron.	Swedish Transformer Iron.
Number of turns of core	4	5 turns
Density of iron (after experiments)	7·775	7·461
Total sectional area of iron	·131	·143 cm.^{2}
Mean diameter of ring	2·35	2·23 cm.
Number of turns of primary	28	29 turns.
,, per cm. of magnetic circuit	3·792	4·143 turns per cm.
Field due to 1 amp. in primary	4·765	5·206 C.G.S. units.
Mean sectional area enclosed by primary.	·72	·83 cm.^{2}
Number of turns of secondary	15	17 turns.
Resistance of iron at 18°	·0486	·0477 ohm.

Electrical Arrangements.—Reduction of Results to Absolute Measure.—The various electrical circuits communicating with the ring-magnet are indicated in fig. 6, which needs little explanation, except perhaps with regard to the arrangement for standardizing the ballistic galvanometer.

For this purpose an auxiliary ring-magnet of known dimensions and having a non-magnetic core was used. Its secondary, in series with that of the iron ring-magnet, formed, in circuit with the galvanometer and a coil of 100 ohms, the secondary circuit, as it remained throughout the experiments. On reversing a known current in the primary of the auxiliary coil, the resulting throw of the ballistic galvanometer was a measure of its sensitiveness to induction alterations in that particular secondary circuit, whether taking place in the core of the auxiliary coil, or in that of the iron ring-magnet. Such standardizing throws were observed before the readings at each temperature, and in this way direct compensation was made for alterations in the resistance of the secondary circuit.

Fig. 6.—General Plan of Connexions.



The standardizing throw and the numerical data of the iron ring-magnet being known, it was possible to obtain from each ballistic observation the change of magnetic induction in the iron core to which it was due; correction being of course made for the induction which did not pass through the core.

The magnetizing current was measured with a tangent galvanometer, whose indications were compared with those of a standard Weston ammeter. The magnetizing force was assumed to be uniform in intensity throughout that part of the section of the ring occupied by the iron; which assumption is, in the present case, but approximately true, though the oblong cross-section favours the uniformity.

Description of the Experiments.

The iron cores in their initial condition may be regarded as unannealed; for the original annealing was destroyed during their preparation.

To study the effects of annealing, measurements were taken during the first heating of specimen A. With this exception, however, all the experiments recorded in this paper were made during the cooling from some definite temperature at which the iron had been thoroughly annealed, that all effects other than those due directly to change of temperatures might be eliminated. The duration of each of the long sets of experiments was about 20 hours.

The ring containing *specimen A* was first heated slowly up to 1050°, stopping at various stages on the way up for observations, and then allowed to cool, when another series of observations was made. It was afterwards heated to 920° to find whether the annealing was complete.

The ring was then taken to pieces, that the cross-section of its iron and the constants of the thermometer-wire might be re-determined.

The arrangements for absorbing the oxygen were with this ring not very perfect. The coil for this purpose had but one iron wire instead of two in parallel; and, becoming locally oxidized, the wire fused before it had absorbed all the oxygen. The cross-section of the iron core on removal from the ring and on detaching the oxide was found to have diminished by as much as 8 per cent. Since, however, no change of section occurred (as deduced from the constancy of the resistance and permeability) during the concluding heating to 920°, it may safely be assumed that the oxidation took place previous to the cooling from 1050°; and that the section during subsequent experiments was the same as that measured at their close.

The ring-magnet, of which *specimen B* was the core, was, after preliminary magnetic measurements, heated at once to 840°, a little above the critical point; at this temperature it was kept for about two hours and then allowed to cool. The following day the ring was again heated to this temperature, it being intended to take a series of observations during cooling. The series had, however, to be postponed, owing to an unfortunate occurrence—the sealing-wax over the cork had slightly softened during the prolonged heating, and the excess of outside pressure began to drive the cork slowly into the jar, rendering it no longer air-tight. The heating current was immediately shut off; but, when cold, the iron core was found to have been partially oxidized by the inrush of air, reducing its section about 4 per cent. To prevent a repetition of this occurrence, an indiarubber tube containing flowing water was arranged to keep the neck of the jar cool.

During the second absorption of the oxygen, the iron-wire coil, which had already been used once for this purpose, burnt out. But it was thought that the absorption might be complete, and the experiments were proceeded with. After heating the ring, therefore, a third time to 840°, a series of measurements was made while the temperature was falling.

The ring was then still further annealed by heating to 1150°; and, during cooling, a set of observations like the previous was taken. On their completion, this ring was also taken to pieces that the cross-section of the core might be re-determined and the constants of the thermometer-wire checked. The last absorption of the oxygen does not seem to have been at all complete, for during the last heating the section diminished by 6 per cent.

Thus, in these experiments, though the method of absorbing the oxygen was shown to be efficient, yet troubles due to oxidation were by no means absent. Irrespective of the diminution of the sectional area of the iron, the presence of oxide in the core might affect the results in a variety of ways:—(1) by the shunting effect due to its conductivity, or (2) by impairing the quality of the insulation, both tending to diminish the apparent resistance of the iron; and (3) by having an appreciable permeability, affecting the magnetic measurements.

(1) The shunting effect is quite negligible. (2) Though the inner surface of the mica was altered by contact with the oxide, the insulation was not seriously injured. (3) With regard to the third possibility :—In small fields, such as those used in the present experiments, the magnetic qualities of soft iron are at ordinary temperatures so very much more intense than even those of the magnetic oxide of iron, that no perceptible alteration of the ballistic galvanometer throws could arise from any oxide present. At temperatures above the critical point, the experiments of M. Curie, already cited, have shown that though the susceptibility of that oxide rapidly decreases with rise of temperature, yet it may be comparable with that of iron, which is also very feebly magnetic in the neighbourhood of 1000° C. The author's methods were, however, not adapted to the determination of the magnetic qualities of feebly magnetic bodies, and any determinations by them of the permeability of the iron core when $\mu < 10$ have in any case little or no worth ; so that errors of such values need hardly be considered.

The only serious effect of the oxidation was to alter the sectional area of the iron. By directly measuring this section both before and after the experiments with each ring-magnet, uncertainties in the absolute values, both of the permeability and resistance measurements, were practically eliminated in all the series of observations, except that taken after annealing specimen B at 840° ; and even in this case the alteration of resistance which took place gave the means of determining with but little uncertainty that sectional area which was not directly measured. Whilst cooling from 1150° a pause of half-an-hour was made at 1071°. During this time the resistance of the iron did not increase, and it is therefore concluded that from this time onwards no further change of section due to oxidation took place.

Method of taking Observations at each Temperature.—The experiments with specimen B were as follows (with specimen A they were similar but not so full) :—

When the temperature had become quite stationary, and the "standardizing throw" had been observed, the iron core was *de-magnetized* by repeatedly reversing the magnetizing current whilst diminishing its strength.

Observations were then taken, by the method of reversals, of the magnetic induction in the core for each of eight values of the field; beginning with $\mathbf{H}=\cdot078$ and $\cdot153$ to get the *permeability under very small forces*, and ending with $\mathbf{H}=9\cdot20$. Next, the iron core was carried through a cyclic process of magnetization between *definite limits of magnetic field* ($\mathbf{H}=\pm6\cdot83$), observations being taken by the "step-by-step" method—sixteen steps round the cycle. This process had, however, been previously gone through several times till the iron had reached a cyclic state. Finally, the iron was carried through a further cyclical magnetization, this time between *definite limits of induction* (*e. g.* $\mathbf{B}=\pm4550$ lines/cm^2).

The last experiment was by far the most difficult; for the magnetizing currents had to be so adjusted that the *sum* of the ballistic throws during one half of the cycle might add up to a certain previously calculated amount depending on the magnitude of the standardizing throw. The difficulty was much reduced by diminishing the number of observations, though the number taken (just eight all round the cycle) barely suffices to determine the hysteresis loop.

The group of experiments at one temperature was concluded by a measurement of the resistance of the iron.

Throughout the whole of such sets of experiments, and particularly when working immediately below the critical point, where the magnetic properties vary so rapidly, the most scrupulous attention was paid to the temperature that it should not, if possible, be allowed to alter more than a fraction of a degree; and this attention was the more necessary when working with the larger magnetizing currents, for their heating effect in the primary circuit was quite perceptible.

The time-period of the ballistic galvanometer needle being about eleven seconds, magnetic "creeping" effects lasting much more than a second would not be recorded.

In reducing the observations for each curve of cyclic magnetization, the mean of the corresponding throws taken during the ascending and descending halves of the process has in every case been taken; though this eliminates any systematic unsymmetry which the curve might have possessed.

Discussion of Results.

The results of the experiments are given in the accompanying Tables and Curves (Plates V. & VI. and Woodcuts, pp. 170–177); the series being in each case arranged in the order in which they were taken.

I. *Curves of Magnetization by the Method of Reversals.*—The effect of annealing (at 1050°) on specimen A is strikingly shown in the curves marked 1, and in the corresponding permeability curves 9 and 10. Its effect on the *maximum* permeability and corresponding field is shown in Table IX. The ultimate result of the annealing was to increase the maximum permeability at ordinary temperatures three-fold ($\mu=1355$ to 4050) whilst reducing the corresponding field to little more than half its former value (3·72 to 2·02).

The first annealing of specimen B affected the permeability in a similar manner (compare curves No. 3). The experiments on this specimen, however, were carried out with a view to showing especially the *difference between annealing at a red heat and at a low white heat.* The curves of magnetization at different temperatures obtained after annealing respectively at 840° and 1150° are numbered 7 and 8; whilst the corresponding permeability-temperature curves for different strengths of magnetic field are shown in the curves 11, 12, and 13.

A reference to these will show that the difference between annealing at these two temperatures is very marked; the iron having by the last annealing been rendered much more susceptible to weak magnetic fields. Annealed at 840°, the maximum permeability at ordinary temperatures was about 4000 in a field of 1·84 C.G.S. units; annealed at 1150° this rose to the remarkably high value of 4680 in the much smaller field of 1·48. The effect of this difference in annealing is, at temperatures immediately below the critical point, even more striking, as is shown by a comparison of the μ-t curves (Nos. 11 & 12) for values of **H** below unity.

An interesting point with regard to specimen B is the *diminution of permeability* which it experiences *in the neighbourhood of* 550° C. This diminution can be traced on the μ-t curves for all but the highest values of **H**, though it is especially noticeable when the magnetizing force is about 1 C.G.S. unit. The effect is more pronounced after the annealing at 1150° than after that at 840°. This peculiarity is hardly, if at all, possessed by specimen A, though it must be borne in mind that the temperature-interval between observations is here much greater, and the effect may thus be disguised. In some earlier experiments made on a ring-magnet having soft commercial iron wire for its core, the minimum of permeability, which occurred at about 500°, was still more marked than in specimen B; the depression being noticeable in the μ-t curves for fields as large as 5 C.G.S. units. It may be mentioned that, in a list of those temperatures at which the physical properties of iron undergo marked alteration, Tomlinson* gives 550° as that at which the internal friction of soft iron begins to rapidly increase, and at which the "specific heat of electricity" for that material changes sign.

The *permeability under small magnetizing forces* ($\mathbf{H} < {\cdot}5$ C.G.S. unit) rises at first slowly (·2 to ·3 per cent. per degree) when the temperature is raised; then, in the neighbourhood of 300°, quite rapidly. It remains nearly constant between 400° and 550°, after which it rises with increasing rapidity to an enormously high value ($\mu = 12660$ for $\mathbf{H} = {\cdot}078$ is the highest *observed*) only to sink still more rapidly at the critical temperature to a value quite insignificant.

As the magnetic field gets smaller (curves 11 and 12) the maximum of permeability increases, and the temperature at which it occurs approaches the critical temperature, so that in very small fields the fall of the permeability is exceedingly sudden.

An examination of the **B-H** curves at various temperatures (Nos. 6, 7, and 8) shows that as the temperature rises, the initial slope of the curve increases and approaches more and more nearly to the maximum slope. And the critical temperature occurs just when they would coincide, *i. e.* when the **B-H** curve would start from the origin with its maximum slope, and the maximum permeability would occur in an infinitely small magnetic field.

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* H. Tomlinson, Phil. Mag. vol. xxvi. July 1888, p. 21.

The *critical temperature* is as near as can be given 795° for specimen A, and 780° for specimen B. For the soft iron wire specimen it was about 770°.

Above this temperature the induction is proportional to the magnetizing force (curves 4 and 5) indicating the *same permeability for all forces*, a result in accordance with the experiments of M. Curie. The curve of variation of the permeability with temperature (No. 13, II.) is given with some diffidence as the observations were liable to considerable error (see first part of Tables IV. and V.), and as the results are too remarkable to pass without further corroboration. It indicates a maximum of about $\mu = 100$ at 820°, and another of $\mu = 17$ at about 1050°; whilst between 920° and 980° the permeability is but 2·3.

II. *Curves of Cyclic Magnetization.*—The variations in the form of these curves due to changes of temperature can be studied in the series of curves Nos. 2, 4, and 5, in which the results recorded in Tables II., VI.–VIII. are plotted.

The process of annealing of specimen A is graphically depicted in the first of these series.

In series No. 5 it will be noticed how, as the temperature rises and approaches 550°, the inclination of the curves to the vertical axis increases, and how with further rise of temperature the curves become more erect and angular. Those taken at 764°·5 in this series are interesting as showing the magnetic condition of the iron almost immediately below the critical temperature. The coercive force required to remove the remanent magnetism was little more than $\frac{1}{10}$ of a C.G.S. unit. For the steep-sloping part of the curve the value of $\frac{d\mathbf{B}}{d\mathbf{H}}$ is about 15000; when $\mathbf{H}$ is much above 1 unit, its value is about 150. The transition from the steep part of the curve to the gently-sloping part is very sudden.

These values of $\frac{d\mathbf{B}}{d\mathbf{H}}$ are of the same order of magnitude in all the sets of curves of both series taken within 50° of the critical temperature, and they seem to represent an approximately constant property of the iron in this region.

The area enclosed by each hysteresis loop has been summed up and divided by 4π, giving the quantity $\frac{1}{4\pi}\int \mathbf{H}\,d\mathbf{B}$, the work done in ergs per c.cm. of the iron by the magnetic forces during the performance of one cycle. The results are given beside the corresponding temperatures in the Tables numbered X. (*a* and *b*), and are plotted in curves 14–16, which show the way in which the hysteresis diminishes to zero as the iron is heated up to the critical temperature.

No evidence of hysteresis above this temperature could be experimentally obtained, a result which one might be led to expect from the fact that the permeability at these temperatures is the same in all magnetic fields.

A comparison of the two curves No. 15 shows the marked diminution of hysteresis at all temperatures which resulted from annealing at 1150° instead of at 840°. The curves expressing the hysteresis for cycles within definite limits of induction (Curves 16 I. & II.) are not comparable, as the limits during the later series were, owing to oxidation, greater than during the former. In order to render them comparable a third curve has been dotted in, for which the ordinates of I. were multiplied by a constant 1·11 (the assumed ratio of the hysteresis when $\mathbf{B} = \pm 4260$ to that when $\mathbf{B} = \pm 4550$, which, according to results of many observers, cannot at ordinary temperatures be far wrong). Comparing then curve I. with curve III. an idea is obtained of the extent to which the hysteresis can be diminished by annealing at the high temperature 1150°, instead of at the lower one 840°. The comparison is given on Table X. *c*.

Table X. *d*. shows concisely the various stages of annealing of specimen B. The hysteresis at ordinary temperatures for $\mathbf{B} = \pm 4550$ was originally 1480 ergs per cub. centim. per cycle. Prolonged annealing at 840° reduced this to about 800 ergs; whilst annealing at 1150° brought the hysteresis down to 612 ergs.

In a paper read before the Institution of Civil Engineers about a year ago*, Prof. Ewing described some tests which he made on a sample of transformer plate of similar thickness and obtained from the same firm as the author's were. The maximum permeability was about 4500, while the hysteresis varied from 740 down to 580 ergs for **B** $= \pm 4000$; and attention is drawn to these values as representing the best iron both as regards high permeability and low hysteresis which he had tested. It is therefore of interest to note that specimen B had after annealing at 1150° a maximum permeability of over 4600 at ordinary temperatures, and a hysteresis as calculated for **B** $= \pm 4000$ of just under 500 ergs, corresponding to a hysteresis loss of but ·295 watts per lb. at a frequency of 100 ∼. The following values are taken from the magnetization curve at 24° C.:—

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* Ewing, "Magnetic Testing of Iron and Steel," Journ. Inst. C. E. May 1896.

B.	**H.**	μ.
2000	·74	2700
4000	1·01	3960
6000	1·33	4520
8000	1·82	4400
10000	2·84	3520
12000	5·63	2130

Between 0° and 200° the *diminution of hysteresis with rise of temperature is* ·08 *per cent. per degree*, when the limits of induction are kept constant.

W. Kunz [*] concludes from his experiments that for soft iron the hysteresis is expressible as a linear function of the temperature. This is not in accordance with the author's results. The curves expressing this relation (Nos. 16) for specimen B are far from straight, and show a decided dip at about 500° corresponding to the minima which occur near that temperature on the permeability-curves.

III. *Resistance-Temperature Curves.*—The variations of specific resistance with temperature are shown graphically on curves 17 and 18.

By taking a sufficient number of observations (see Tables XI. and XII.) the author has been able to show that the resistance-temperature curve undergoes *no abrupt change of direction* at or near the critical temperature. (Previous observers have given curves all of which were *broken* at this temperature; an examination of the figures given by Hopkinson and Le Chatelier in the papers already cited shows that in each case they would be more strictly satisfied by a *continuous curve* like those here given.)

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[*] W. Kunz, *Elektrotechnische Zeitschrift*, 1894, p. 196.

The annealing of specimen A has reduced its specific resistance considerably.

With specimen B, the further annealing at 1150° has but slightly altered it from the condition as regards specific resistance in which it was after the annealing at 840°.

The temperature-coefficients and specific resistance at 0° and 1000° are given at the foot of Tables XI. and XII.

As regards the resistance-changes both of specimens A and B, and of the soft commercial iron previously tested :—The resistance rises with rising temperature at a rate which increases fairly uniformly till a specific resistance of about 100,000 C.G.S. units is reached. Here the maximum slope of the curve occurs at a point *but few degrees below the critical temperature.* The rate of increase of resistance then falls off rapidly, so that at 1000° the temperature-coefficient is about ·0024 and the specific resistance 118,000.

With specimen B, which, judging from the low value of its specific resistance at 0° (10,050 C.G.S. units) and from the high value (·0057 at 0°) of its temperature-coefficient, should be a sample of very pure iron, the *maximum coefficient* after annealing at 1150° was ·0204 at about 765°. The number of observations on this resistance-temperature curve (18 II.) make it perhaps permissible to note its irregularities a little more closely. A flat part of the curve will be noticed between 830° and 900°, and the temperature-coefficient, which remains tolerably constant over this range, has at 865° the value ·0068. (Dr. Hopkinson gave ·0067 as the coefficient of iron above the critical temperature.) The points of maximum curvature 800° and 920° seem to be in agreement with the minima of the permeability curve (No. 13) obtained at the same time.

The above experiments were carried out in the laboratories of the Physical Institute of the Polytechnicum, Zürich ; and the author takes this opportunity of expressing his gratitude to the director, Prof. H. F. Weber, for encouragement received throughout the progress of the work.

Zürich, March 1897.

TABLE I.—SPECIMEN A.

Induction and Permeability in various Magnetic Fields taken at different temperatures.

(*a*) Before annealing. (Curves 1 and 9.)

H in C.G.S. units.	20°.		220°.		365°.		532°.		660°.	
	B	μ	**B**	μ	**B**	μ	**B**	μ	**B**	μ
·917	401	438	463	505	506	552	933	1017	1864	2033
2·24	2495	1114	2868	1281	2990	1335	4180	1867	5285	2360
3·94	5325	1352	5345	1357	5145	1306	6380	1620	7215	1830
6·10	7465	1225	7270	1192	6850	1107	8000	1312	8685	1440
8·26	8805	1066								

(*b*) After annealing at 1050° C. (Curves 1, 6, and 10.)

H in C.G.S. units.	774°.		722°.		636°.		506°.	
	B	μ	**B**	μ	**B**	μ	**B**	μ
·070			292	4170	109	1560	64	915
·140			583	4170	272	1950	166	1190
·343			2475	7220	1188	3465	617	1800
·917	5980	6520	6008	6550	5395	5880	4790	5220
2·24	7795	3480	8360	3730	8630	3850	8820	3940
3·94	8290	2105	9475	2405	9870	2480	10320	2620
6·10	8330	1366	10180	1670	10705	1750	11310	1855

H in C.G.S. units.	324°.		188°.		21°·5.	
	B	μ	**B**	μ	**B**	μ
·070	48	685	29	414	21 ?	300 ?
·140	104	743	68	485	50	357
·343	366	1067	248	723	181	527
·917	3355	3660	1890	2060	1208	1318
2·24	9105	4060	9175	4095	8845	3950
3·94	10925	2775	11420	2900	11435	2904
6·10	11960	1960	12550	2060	12770	2093

(*c*) After re-heating to 920°. (Curves not given.)

H in C.G.S. units.	770°.		759°.		680°.		630°.		22°.	
	B	μ	**B**	μ	**B**	μ	**B**	μ	**B**	μ
·070			158	2260	112	1600	69	985	24	343
·140			391	2800	261	1870	164	1170	53	379
·343			1908	5560	1298	3785	810	2360	168	490
·917	5810	6340	5850	6380	5745	6260	5020	5470	1430	1560
2·24	8045	3590	8120	3625	8560 ?	3820 ?	8390	3745	8485	3790
3·94	8940	2270	9465	2405	9945	2525	9825	2495	11380	2890
6·10	9110	1494	10180	1670	10840	1778	10790	1770	12790	2095

TABLE II.—Specimen A.

Data for Cyclic **B-H** *Curves at various Temperatures.*
Limits of Cycle: **H** = ±6·10.

(*a*) Before annealing. (Curves 2.)

H	20°.	224°.	367°.	534°.	665°.
	B	**B**	**B**	**B**	**B**
6·10	7370	7145	6830	7980	8610
3·94	6985	6645	6300	7365	7715
2·24	6380	5945	5530	6500	6570
·917	5620	5090	4660	5410	5240
0	4800	4232	3805	4265	3750
−·917	3513	2793	2285	1750	−526
−2·24	−1642	−2365	−2600	−3935	−5148
−3·94	−5410	−5345	−5170	−6380	−7140
−6·10	−7370	−7145	−6830	−7980	−8610
Hysteresis in ergs per cm.³	3900	3210	2800	2670	1850

(*b*) After annealing at 1050° C. (Curves 2 and 14.)

H	765°.	708°.	625°.	506°.	325°.	175°.	21°·5.
	B	**B**	**B**	**B**	**B**	**B**	**B**
6·10	9100	10220	10640	11185	11850	12320	12480
3·94	8790	9630	10085	10615	11400	12000	12190
2·24	8075	8720	9220	9880	10815	11520	11760
·917	6720	7205	7730	8565	9925	10835	11180
0	2985	3930	4780	6190	8170	9565	10210
−·917	−5830	−5845	−5290	−4290	−2005	1389	4170
−2·24	−7880	−8275	−8400	−8620	−9090	−9145	−8990
−3·94	−8780	−9440	−9750	−10190	−10810	−11210	−11340
−6·10	−9100	−10220	−10640	−11185	−11840	−12320	−12480
Hysteresis in ergs per cm.³	500	920	1345	2060	3090	4110	4570

(*c*) After re-heating to 920°. (Curves not given.)

H	738°.	679°.	630°.	22°.
	B	**B**	**B**	**B**
6·10	10370	10840	10660	12520
3·94	9565	10115	10020	12210
2·24	8515	9120	9085	11790
·917	6880	7510	7505	11185
0	3535	4397	4500	9995
−·917	−5610	−5240	−4435	2766
−2·24	−8100	−8365	−8080	−8420
−3·94	−9450	−9820	−9670	−11210
−6·10	−10370	−10840	−10660	−12520
Hysteresis in ergs per cm.³	780	1280	1510	4720

TABLE III.—Specimen B.

Preliminary Measurements at 24° C. (Curves No. 3.)

Induction and Permeability in various Magnetic Fields.

H	Initial condition.		After 2 hours at 840°.		After 5 hours at 840°.	
	B	μ	**B**	μ	**B**	μ
·078	14	178	32	410	35	450
·153	37	242	79	516	84	549
·378	102	270	280	741	307	813
1·017	560	550	2656	2608	2995	2942
2·49	4352	1746	10295	4135	10525	4230
4·40	7265	1652	12395	2816	12405	2820
6·83	9220	1349	13420	1963	13340	1953
9·20	10420	1133	14005	1523	13920	1515

Cyclic **B-H** Curves between Limits $\mathbf{H} = \pm 6{\cdot}83$.

H	Initial condition.	After 2 hours at 840°.	After 5 hours at 840°.
	B	**B**	**B**
6·83	9120	13100	13120
4·40	8670	12800	12820
2·49	8005	12425	12430
1·017	7135	11905	11885
0	6165	10890	10740
−1·017	4265	1420	778
−2·49	−3882	−10105	−10380
−4·40	−7270	−12140	−12210
−6·83	−9120	−13100	−13120
Hysteresis in ergs per c. cm.	4740	5020	4625

Cyclic **B-H** Curves at 20° C.

Limits of Cycle: $\mathbf{B} = \pm 4020$.

Initial condition.	
H	**B**
2·47	4025
1·007	3520
0	2763
−1·007	870
−2·47	−4025
Hysteresis per cycle	$= 1218 \frac{\text{ergs}}{\text{cm.}^3}$

After 2 hours at 840°.	
H	**B**
? 1·15	4015
·646	3760
0	2887
− ·646	− 63
−1·15	−4015
Hysteresis $= 664 \frac{\text{ergs}}{\text{cm.}^3}$	

Limits $\mathbf{B} = \pm 4140$.

After 5 hours at 840°.	
H	**B**
1·170	4140
·660	3810
0	2900
− ·660	− 242
−1·170	−4140
Hysteresis $= 669 \frac{\text{ergs}}{\text{cm.}^3}$	

TABLE IV.—Specimen B.

Induction and Permeability in various Magnetic Fields taken successively during cooling from 840° C.

Above critical Temperature. (Curves 4, 7, and 13.)

H	820°.		816°.		799°.	
	B	μ	**B**	μ	**B**	μ
1·017			19	19	40	39
2·49	29	12	35	14	97	39
4·40			78	18	163	37
6·83	84	12	101	15	262	38
9·20			124	13	363	39
Mean Permeability }		12	...	16	...	38

H	797°.		788°.		784°·5.		782°·5.	
	B	μ	**B**	μ	**B**	μ	**B**	μ
1·017					124	122		
2·49					273	110		
4·40					518	118		
6·83	349	51	409	60	787	115	825 ?	121 ?
9·20					1035	113		
Mean Permeability }		...	...	...	...	116	...	...

Below critical Temperature. (Curves 4, 7, and 11.)

H	770°.		764°.		746°.	
	B	μ	**B**	μ	**B**	μ
·078			298	3820	198	2540
·153			663	4330	490	3200
·378			2780	7360	2248	5950
1·017			5675	5575	5830	5730
2·49			6720	2700	8020	3220
4·40	5960	1355	7070	1606	8565	1950
6·83	6500	952	7410	1084	8905	1304
9·20			7870	856	9165	996

H	710°.		648°·5.		567°.		467°.	
	B	μ	**B**	μ	**B**	μ	**B**	μ
·078	131	1680	90	1150	61	780	57	730
·153	322	2100	230	1500	141	920	143	935
·378	1722	4560	1041	2755	645	1710	511	1354
1·017	5955	5850	5685	5585	5110	5020	5195	5100
2·49	8630	3470	9030	3625	8995	3610	9475	3800
4·40	9840	2260	10440	2375	10695	2430	11030	2510
6·83	10580	1550	11350	1662	11770	1725	12090	1770
9·20	10955	1191	11870	1290	12315	1340	12660	1377

Table IV. (*continued*).

H	360°·5.		251°·5.		136°.		24°.	
	B	μ	B	μ	B	μ	B	μ
·078	50	640	31	400	30	380	20	250
·153	126	820	70	450	68	440	40	260
·378	571	1510	294	778	318	840	178	470
1·017	5125	5040	4360	4280	3253	3200	1934	1900
2·49	9800	3935	9940	3990	10005	4020	9420	3780
4·40	11580	2630	11980	2720	12140	2760	11835	2690
6·83	12705	1862	12890	1888	13260	1942	12980	1902
9·20	13320	1449	13525	1471	13860	1507	13600	1480

TABLE V.—SPECIMEN B.

Induction and Permeability in various Magnetic Fields taken successively during cooling from 1150° C.

Above critical Temperature. (Curves 5, 8, 12, and 13.)

H	1145°.		1047°.		980°.		975°.		960°.	
	B	μ	B	μ	B	μ	B	μ	B	μ
1·017										
2·49			42	17	15	6				
4·40										
6·83	52 ?	8					17	2·5	17	2·5
9·20										
Mean Permeability ...	...	8	...	17	...	6	...	2·5	...	2·5

H	922°.		895°·5.		851°.		829°·5		816°.		787°·5.	
	B	μ	B	μ	B	μ	B	μ	B	μ	B	μ
1·017									108	106	51	50
2·49			129	52	173	69	260	104	260	104	120	48
4·40							434	99	434	99	215	49
6·83	16	2·3	388	57	475	70	650	96	781	114	336	49
9·20											518?	56?
Mean Permeability	}	2·3	...	54	...	69·5	...	100	...	106	...	51

Below critical Temperature. (Curves 5, 8, and 12.)

H	776°·5.		775°.		764°·5.		748°.	
	B	μ	B	μ	B	μ	B	μ
·078					631	8090	485	6210
·153					1935	12660	1585	10360
·378					4210	11140	4465	11820
1·017	925	908	1353	1330	5020	4930	6720	6600
2·49	1788	718	1964	789	5780	2320	7980	3200
4·40			2442	556	6305	1434	8290	1885
6·83			2798	410	6440	943	8555	1252
9·20			3006	327	6550	712	8645	940

Table V. (*continued*).

H	730°.		695°.		634°.		554°.	
	B	μ	**B**	μ	**B**	μ	**B**	μ
·078	372	4770	243	3120	126	1620	73	935
·153	1161	7600	713	4660	332	2170	172	1120
·378	4165	11020	3565	9430	2128	5630	996	2640
1·017	6850	6730	6845	6720	5780	5680	4275	4200
2·49	8415	3380	8985	3610	8800	3530	7565	3040
4·40	9160	2085	9985	2270	10160	2280	9630	2190
6·83	9495	1390	10670	1562	11050	1620	10940	1603
9·20	9730	1024	11110	1209	11670	1269	11690	1271

H	457°.		352°.		249°.		137°·5.		24°.	
	B	μ	**B**	μ	**B**	μ	**B**	μ	**B**	μ
·078	78	1000	77	980	48	615	43	550	35	450
·153	172	1125	167	1090	113	740	99	650	85	555
·378	989	2620	879	2325	457?	1210?	471	1246	356	942
1·017	5200	5110	5115	5020	4730	4650	4410	4330	4010	3940
2·49	8710	3495	9230	3705	9385	3770	9740	3910	9480	3810
4·40	10460	2380	10930	2485	11145	2535	11560	2625	11400	2590
6 83	11475	1680	11930	1748	12170	1783	12495	1830	12440	1823
9·20	12150	1321	12460	1355	12760	1388	13090	1423	13040	1418

TABLE VI.—SPECIMEN B.

Data for Cyclic **B-H** *Curves* taken successively at diminishing temperatures after annealing at 840° C.

Limits of Cycle : **H** $= \pm 6{\cdot}83$. (Curves 4 and 15.)

H	764°.	746°.	710°.	648°·5.	567°.
	B	**B**	**B**	**B**	**B**
6·83	7370	8870	10350	11130	11495
4·40	7060	8545	9730	10500	10935
2·49			8870	9665	10090
1·017	6005	6425	7380	8295	8620
0		3246	4330	5515	5770
−1·017	−5560	−5820	− 6000	− 5555	− 4535
−2·49			− 8555	− 8900	− 8860
−4·40	−7060	−8480	− 9700	−10250	−10480
−6·83	−7370	−8870	−10350	−11130	−11495
Hysteresis in ergs per c. cm.	426	573	1010	1650	2180

Table VI. (*continued*).

H	467°.	360°·5.	251°·5.	136°.	24°.
	B	**B**	**B**	**B**	**B**
6·83	11910	12540	12770	13040	12685
4·40	11400	12110	12390	12720	12360
2·49	10695	11495	11805	12230	11895
1·017	9520	10355	10810	11380	11110
0	7240	8245	8925	9770	9680
−1·017	− 4940	− 4010	− 2608	− 429	1395
−2·49	− 9340	− 9715	− 9820	− 9910	− 9395
−4·40	−10830	−11495	−11735	−11980	−11610
−6·83	−11910	−12540	−12770	−13040	−12685
Hysteresis in ergs per c. cm.	2550	3195	3700	4360	4700

TABLE VII.—SPECIMEN B.

Data for Cyclic **B-H** *Curves* taken successively at diminishing temperatures after annealing at 1150° C.

Limits of Cycle: **H** = ± 6·83. (Curves 5 and 15.)

H	764°·5.	748°.	730°.	695°.	634°.
	B	**B**	**B**	**B**	**B**
6·83	6145	8240	9420	10615	10760
4·40	5815	8030	9100	10035	10060
2·49	5490	7670	8495	9205	9080
1·017	5075	6800	7185	7820	7180
0	1389	2455	2856	3480	3215
−1·017	−5060	−6540	−6835	− 6920	− 5490
−2·49	−5485	−7575	−8320	− 8870	− 8430
−4·40	−5795	−7975	−9040	− 9850	− 9860
−6·83	−6145	−8240	−9420	−10615	−10760
Hysteresis in ergs per c. cm. per cycle	120	328	426	797	1010

H	554°.	457°.	352°.	249°.	137°·5.	24°.
	B	**B**	**B**	**B**	**B**	**B**
6·83	10825	11360	11735	12110	12280	12150
4·40	10025	10810	11230	11730	11950	11810
2·49	8650	9950	10435	11125	11420	11300
1·017	6377	8340	8875	9980	10510	10400
0	3322	5395	6050	7520	8525	8560
−1·017	− 4140	− 4930	− 3960	− 3881	− 3196	− 2327
−2·49	− 7600	− 8710	− 8650	− 9295	− 9570	− 9290
−4·40	− 9575	−10365	−10620	−11100	−11330	−11150
−6·83	−10825	−11360	−11735	−12110	−12280	−12150
Hysteresis in ergs per c. cm. per cycle	1345	2025	2565	3130	3500	3660

TABLE VIII.—SPECIMEN B.

Data for Cyclic **B-H** *Curves between constant limits of Induction* taken successively at diminishing temperatures.

(*a*) After annealing at 840° C.

Limits of Cycle : **B** = ± 4260. (Curves 4 and 16.)

764°.		746°.		710°.		648°·5.		567°.	
H	**B**	**H**	**B**	**H**	**B**	**H**	**B**	**H**	**B**
·60	4250	·60	4255	·73	4265	·81	4260	·93	4250
·31	3520	·32	3547	·34	3668	·36	3725	·56	3970
0	2092	0	2261	0	2611	0	2860	0	2870
−·31	−1762	−·32	−1359	−·34	− 440	−·36	− 189	−·56	−2263
−·60	−4250	−·60	−4255	−·73	−4265	−·81	−4260	−·93	−4250
Hysteresis per cycle	$=187 \frac{\text{ergs}}{\text{cm.}^3}$	$216 \frac{\text{ergs}}{\text{cm.}^3}$		$318 \frac{\text{ergs}}{\text{cm.}^3}$		$357 \frac{\text{ergs}}{\text{cm.}^3}$		$398 \frac{\text{ergs}}{\text{cm.}^3}$	

467°.		360°·5.		251°·5.		136°.		18°.	
H	**B**	**H**	**B**	**H**	**B**	**H**	**B**	**H**	**B**
·88	4265	·93	4265	1·05	4265	1·20	4265	1·318	4265
·38	3760	·40	3743	·42	3732	·43	3700	·426	3725
0	3010	0	2930	0	3000	0	3033	0	3115
−·38	− 733	−·40	924	− ·42	1378	− ·43	1630	− ·426	1910
−·88	−4265	−·93	−4265	−1·05	−4265	−1·20	−4265	−1·318	−4265
Hysteresis per cycle	$=396 \frac{\text{ergs}}{\text{cm.}^3}$	$495 \frac{\text{ergs}}{\text{cm.}^3}$		$590 \frac{\text{ergs}}{\text{cm.}^3}$		$656 \frac{\text{ergs}}{\text{cm.}^3}$		$744 \frac{\text{ergs}}{\text{cm.}^3}$	

(*b*) After annealing at 1150° C.

Limits of Cycle : **B** = ± 4550. (Curves 5 and 16.)

764°·5.		748°.		730°.		695°.	
H	**B**	**H**	**B**	**H**	**B**	**H**	**B**
·640	4543	·403	4540	·443	4547	·516	4540
·210	3690	·174	3662	·184	3722	·197	3712
0	1660	0	2002	0	2272	0	2564
—·210	− 2665	−·174	−1921	−·184	−1468	− ·197	− 897
−·640	−4543	−·403	−4540	−·443	−4547	−·516	−4540
Hysteresis per cycle	$=81 \frac{\text{ergs}}{\text{cm.}^3}$	$109 \frac{\text{ergs}}{\text{cm.}^3}$		$128 \frac{\text{ergs}}{\text{cm.}^3}$		$178 \frac{\text{ergs}}{\text{cm.}^3}$	

Table VIII. (*continued*).

634°.		554°.		457°.		352°.	
H	**B**	**H**	**B**	**H**	**B**	**H**	**B**
·730	4540	·935	4550	·898	4553	·881	4542
·336	3928	·373	3770	·371	3840	·361	4002
0	2720	0	2520	0	2827	0	3060
−·336	−1886	−·373	−1074	−·371	− 488	−·361	239
−·730	−4540	−·935	−4550	−·898	−4553	−·881	−4542
Hysteresis per cycle } $=264 \frac{\text{ergs}}{\text{cm.}^3}$		$335 \frac{\text{ergs}}{\text{cm.}^3}$		$379 \frac{\text{ergs}}{\text{cm.}^3}$		$475 \frac{\text{ergs}}{\text{cm.}^3}$	

249°.		137°·5.		18°.	
H	**B**	**H**	**B**	**H**	**B**
1·001	4550	1·032	4545	1·114	4545
·382	3952	·388	3980	·398	3972
0	3043	0	3188	0	3184
− ·382	712	− ·388	1163	− ·398	1386
−1·001	−4550	−1·032	−4545	−1·114	−4545
Hysteresis per cycle } $=508 \frac{\text{ergs}}{\text{cm}^3}$.		$555 \frac{\text{ergs}}{\text{cm.}^3}$		613 } ergs per c. cm. per cycle.	

TABLE IX.

Relation of Maximum Permeability and of the Field in which it occurs to Temperature.

(The values are estimated from the curves, and are only *approximate*.)

SPECIMEN A.

(Curves 9 and 10.)

Before annealing.

Temperature.	Maximum Permeability.	Corresponding Magnetic Field.
°C.	μ	**H**
20	1355	3·70
220	1370	3·25
365	1360	2·80
532	1880	2·20
660	2650	1·50

After annealing at 1050°.

Temperature.	Maximum Permeability.	Corresponding Magnetic Field.
°C.	μ	**H**
722	8350	·44
636	5950	·92
506	5450	1·05
324	4860	1·46
188	4370	1·70
21·5	4050	2·02

Table IX. (*continued*).

Specimen B.

(Curves 11 and 12.).

After annealing at 840°.

Temperature.	Maximum Permeability.	Corresponding Field.
°C.	μ	**H**
764	7800	·50
746	7090	58
710	6070	72
648·5	5610	93
567	4900	1·09
467	5190	1·18
360·5	5280	1·13
251·5	4940	1·38
136	4250	1·69
24	3980	1·84

After annealing at 1150°.

Temperature.	Maximum Permeability.	Corresponding Field.
°C.	μ	**H**
764·5	14600	·20
748	13400	·25
730	11700	·33
695	9430	·43
634	6040	·58
554	4240	·99
457	5120	·94
352	5050	·96
249	4820	1·19
137·5	5020	1·36
24	4680	1·48

TABLE X.

Variation of Magnetic Hysteresis with Temperature.

(See Curves 14, 15, and 16.)

(*a*) Cycle between constant Limits of Magnetizing Force.

Specimen A.

Limits: **H** = ±6·10.
After annealing at 1050°.

Temperature.	Hysteresis in ergs per c.cm. per cycle.
°C.	
765	500
708	920
625	1345
506	2060
325	3090
175	4110
21·5	4570

Specimen B.

Limits: **H** = ±6·83.

Annealed at 840°.

Temperature.	Hysteresis in ergs per c.cm. per cycle.
°C.	
764	426
746	573
710	1010
648·5	1650
567	2180
467	2550
360·5	3195
251·5	3700
136	4360
24	4700

Annealed at 1150°.

Temperature.	Hysteresis in ergs per c.cm. per cycle.
°C.	
764·5	120
748	328
730	426
695	797
634	1010
554	1345
457	2025
352	2565
249	3130
137·5	3500
24	3660

Table X. (*continued*).

(*b*) Cycle between constant Limits of Magnetic Induction.

SPECIMEN B.

Annealed at 840°. Limits: $\mathbf{B}=\pm 4260$.

Temperature.	Hysteresis in ergs per c.cm. per cycle.
°C.	
764	187
746	216
710	318
648·5	357
567	398
467	396
360·5	495
251·5	590
136	656
18	744

Annealed at 1150°. Limits: $\mathbf{B}=\pm 4550$.

Temperature.	Hysteresis in ergs per c.cm. per cycle.
°C.	
764·5	81
748	109
730	128
695	178
634	264
554	335
457	379
352	475
249	508
137·5	555
18	613

(*c*) SPECIMEN B.

Comparison of Hysteresis after annealing at 840° and 1150°.

(Reduced to a uniform Induction Limit of $\mathbf{B}=\pm 4550$.)

Temperature.	Annealed at 840°.	Annealed at 1150°.
°C.		
20	822	612
200	686	530
400	504	437
600	430	302
700	358	176
750	236	106

(*d*) Hysteresis of SPECIMEN B at 20° and Induction Limits ±4550, showing various stages of annealing.

	Hysteresis in ergs per c.cm. per cycle.
Initial condition	1480
After 2 hrs. at 840°	810
„ 5 hrs. „	780
„ 6 hrs. „ and *very* gradual cooling.	822
After heating to 1150° and *very* gradual cooling.	612

TABLE XI.

Iron Specimen A.

Electrical Resistance at Different Temperatures.

(*a*) Before and during annealing. (Curve 17, I.)

Temperature. °C.	Specific Resistance in C.G.S. units.	Temperature. °C.	Specific Resistance in C.G.S. units.
18°	14610	530·5°	61550
127	22150	535·5	61550
189	26300	643	77400
210·5	27680	674	83400
213	27880	752	96700
354·5	41300	763	99700
373	43450		

(*b*) After annealing at 1050°. (Curve 17, II.)

[Critical Temp.: 795°.]

Temperature. °C.	Specific Resistance in C.G.S. units.	Temperature. °C.	Specific Resistance in C.G.S. units.
1050°	115800 ?	330·5°	37000
1026·5	115800 ?	201·5	25130
950·5	114200	171·5	23480
936	113100	18	13820
870	109400	16	13580
881	110000		
808·5	101100	After reheating to 920°.	
805	101100	(Also plotted for Curve II.)	
786	97200		
761	92600	885·5	110600
724·5	86600	916	112800
706	84400	736	89000
653	75600	686·5	81300
620	71350	681	79750
504·5	55550	632·5	72400
506	55700	630	72900
327	36760	20	13940
321	36280	16·8	13510

		Before annealing.	After annealing. at 1050°.	
Temperature-Coefficient	at 0°	·0044	·0045	
	maximum value at about 780°	·0125	·0144	
	at 1000°	......	·00236	
Specific Resistance......	at 0°	13600	12500	C.G.S. Units.
	at 1000°	......	115600	

TABLE XII.

IRON SPECIMEN B.

Electrical Resistance at Different Temperatures.

(*a*) After annealing at 840°. (Curve 18, I.)

[Critical Temp. = 782°.]

Temp. ° C.	Sp. Resist. in C.G.S. Units.	Temp. ° C.	Sp. Resist. in C.G.S. Units.	Temp. ° C.	Sp. Resist. in C.G.S. Units.
826°	108300	679°	83750	358°5	39800
800	105100	649	78900	314·5	34920
786·5	103000	605·5	71950	252·5	28660
772	100000	555	64400	190·5	23280
764	98600	561	65400	128·5	18440
751	96200	517	59250	101	16410
739·5	94200	466·5	52700	52	13210
714·5	89300	413·5	45900	18	11110

(*b*) After annealing at 1150°. (Curve 18, II.)

[Critical Temp. = 778°.]

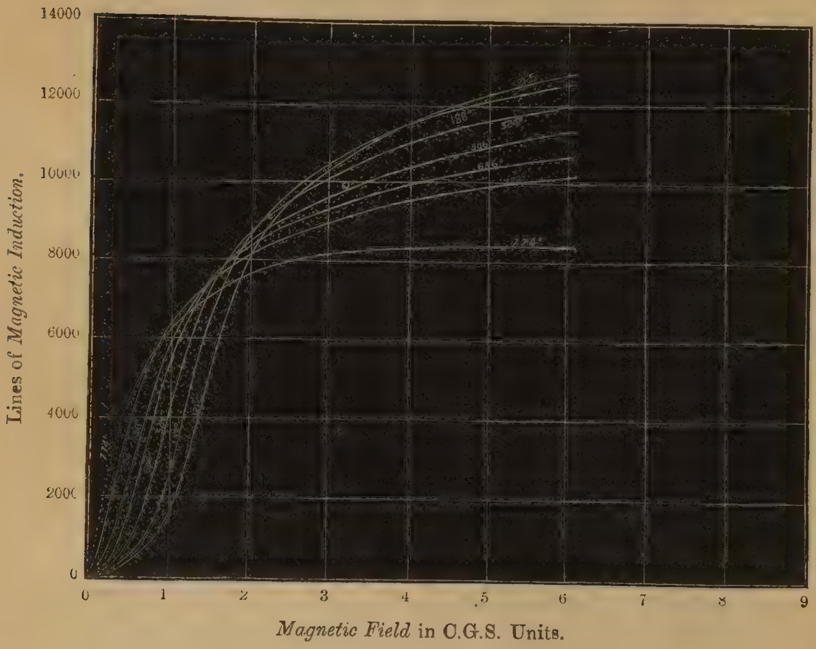
Temp. ° C.	Sp. Resist. in C.G.S. Units.	Temp. ° C.	Sp. Resist. in C.G.S. Units.	Temp. ° C.	Sp. Resist. in C.G.S. Units.
1150°	120600 ?	871°	113400	630°5	76400
1142	121000 ?	858	112500	637	77400
1115·5	120850	844	111400	594	70850
1095·5	120600	824·5	109600	553·5	64850
1071·5	120350	813·5	108400	554	64900
1058·5	120050	792·5	105600	503·5	57850
1036	119650	787	104600	454·5	51770
1014·5	119100	781	103400	406	45720
989·5	118550	770	100900	386	43400
983	118350	765·5	100000	861	40580
977·5	118200	764·5	99850	295·5	34020
972	118100	745·5	96150	252·5	29080
946·5	117200	744	95650	196·5	24020
926·5	116600	725·5	92050	140·5	19410
916·5	116200	697·5	87200	84	15390
891·5	114900	658	80850	18	11140

		After annealing at 840°.	After annealing at 1150°.	
Temperature-Coefficient	at 0°	·0057	·0057	
	maximum value at about 765°	·0189	·0204	
	at 1000°	......	·00244	
Specific Resistance	at 0°	10050	10050	C.G.S. Units.
	at 1000°	......	118800	

B-H *Curves at various Temperatures.*

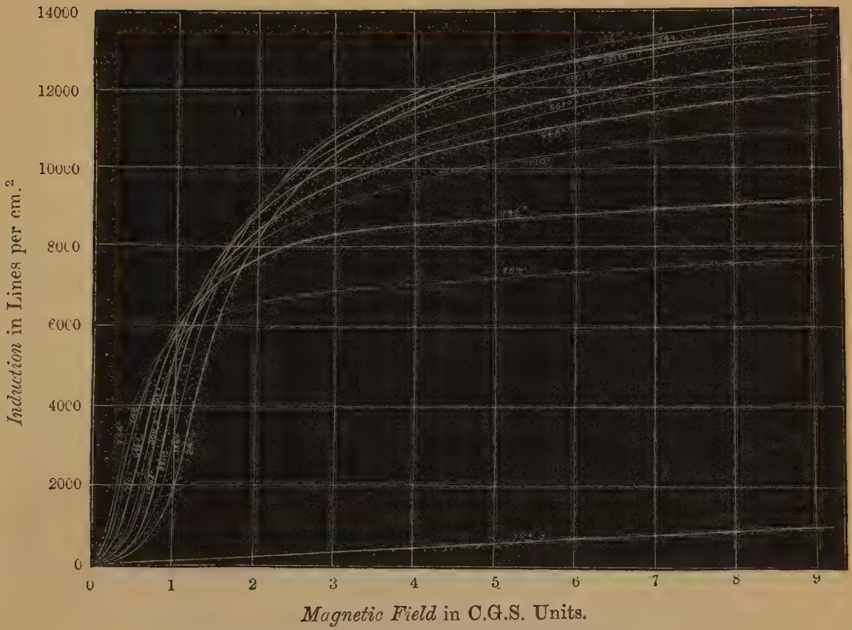
Specimen A after annealing at 1050° C.

6.



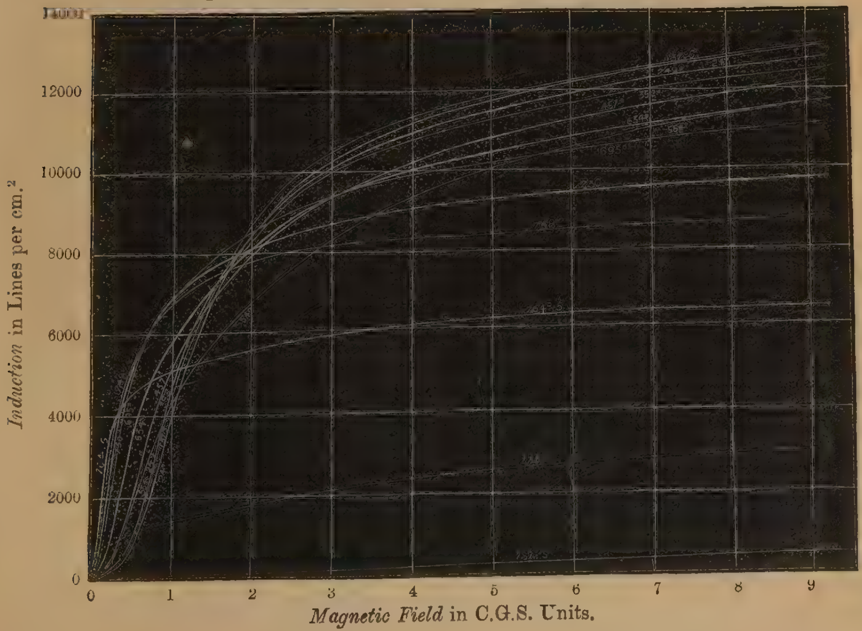
B-H *Curves at various Temperatures.*
Specimen B after annealing at 840° C.

7.



B-H *Curves at various Temperatures.*
Specimen B after annealing at 1150° C.

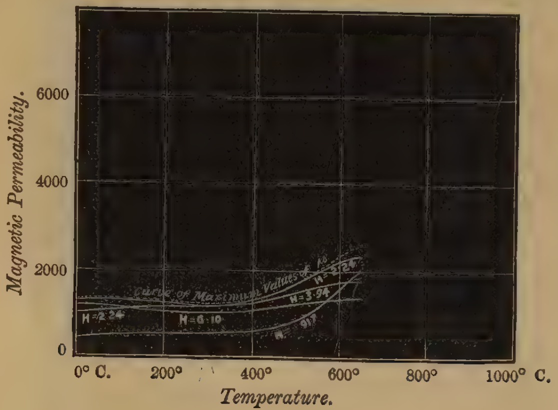
8.



Permeability-Temperature Curves.

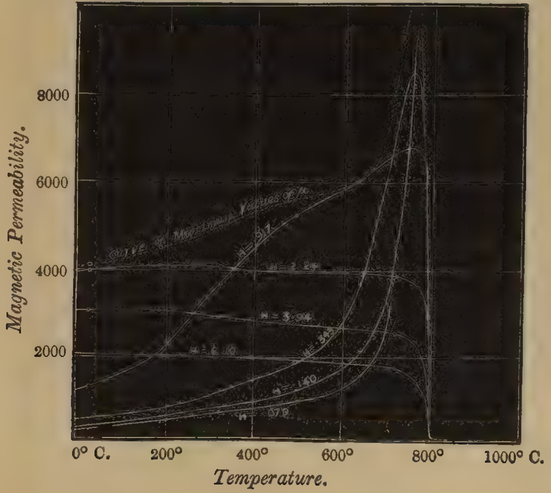
Specimen A before annealing.

9.



Specimen A after annealing at 1050° C.

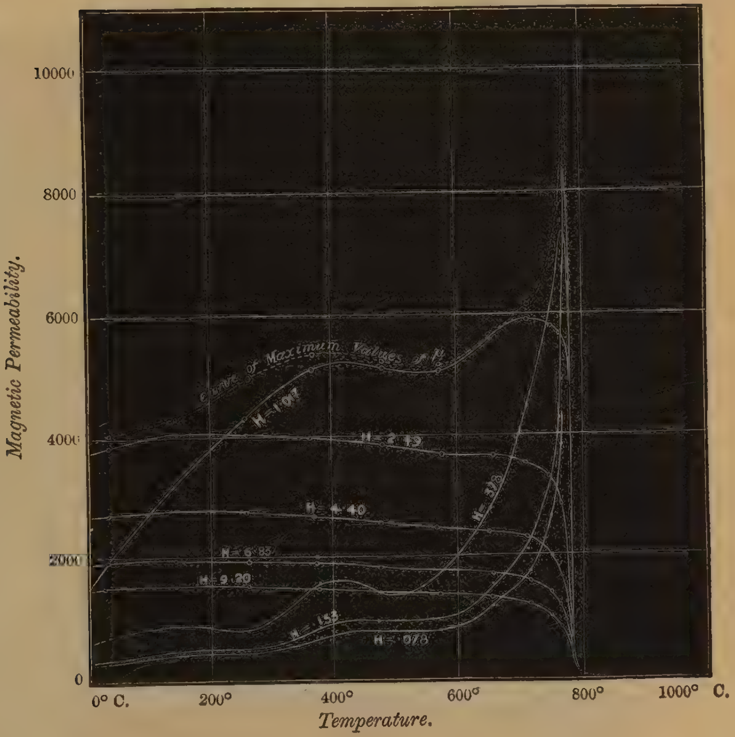
10.



Permeability-Temperature Curves.

Specimen B after annealing at 840° C.

11.

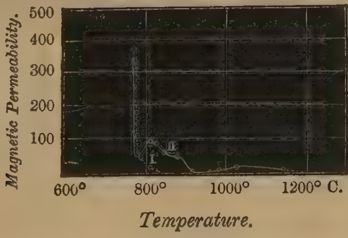


Magnetic Permeability of Iron specimen B

at Temperatures above its Critical Point.

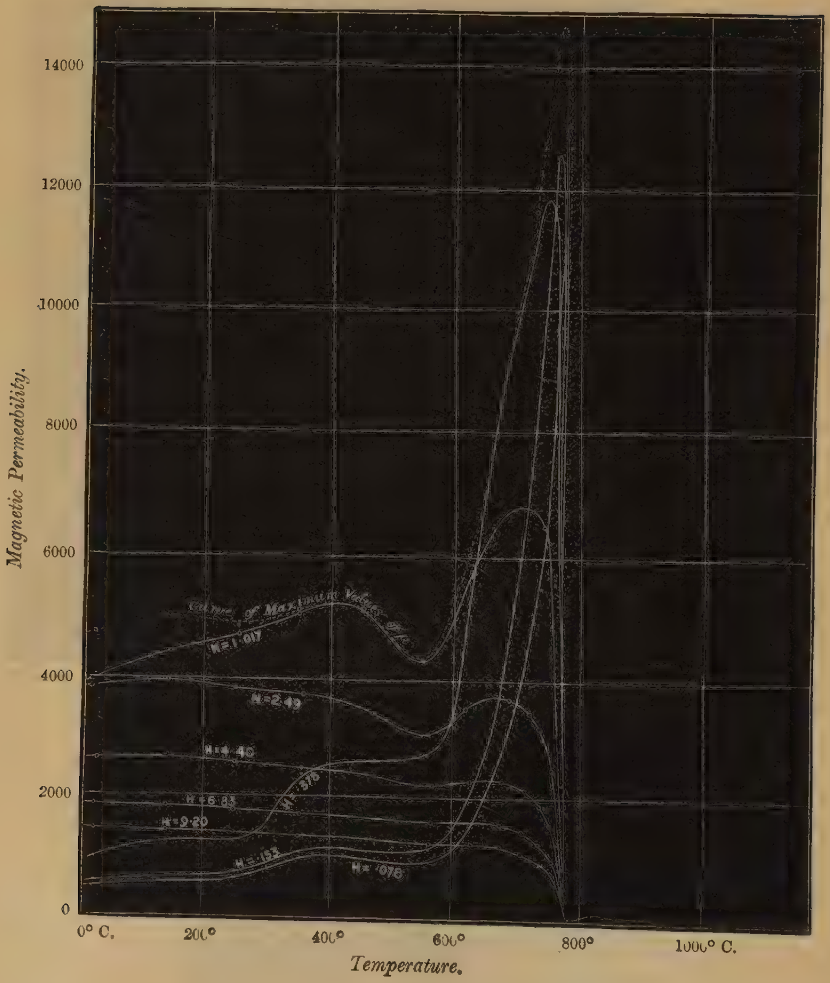
Curve I. After annealing at 840° C. Curve II. After annealing at 1150° C.

13.



Specimen B after annealing at 1150° C.

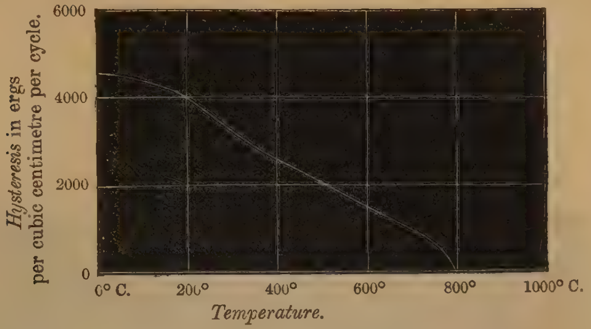
12.



Hysteresis-Temperature Curves.

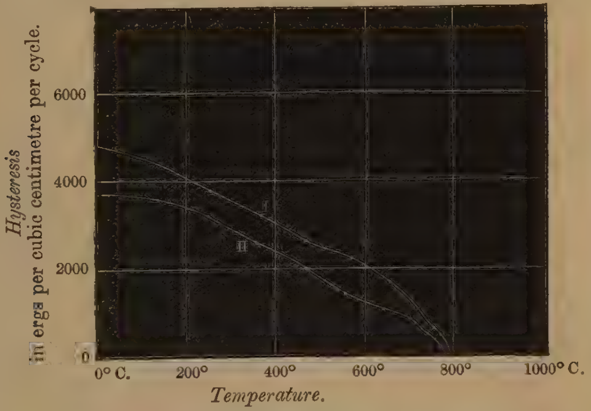
Specimen A. Annealed at 1050° C. Limits of Cycle. **H** = ±6·10.

14.



Specimen B. Limits of Cycle. **H** = ±6·83 C.G.S. Units:
Curve I. After annealing at 840° C. Curve II. After annealing at 1150° C.

15.

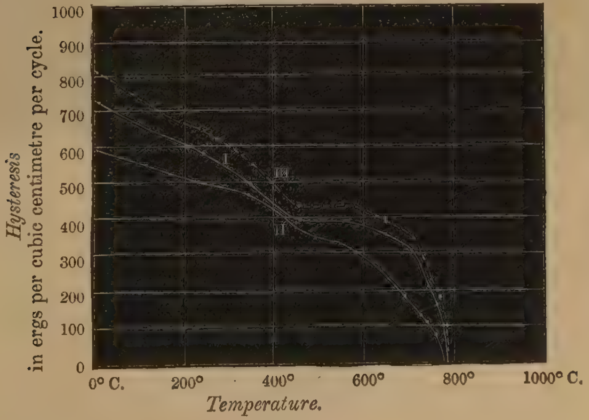


Specimen B.

Limits of Hysteresis Loop.

Curve I.	After annealing at 840° C.	**B** = ±4260.
,, II.	After annealing at 1150° C.	**B** = ±4550.
,, III.	Calculated from I. for the Limits	**B** = ±4550.
	(to compare with II.)	

16.

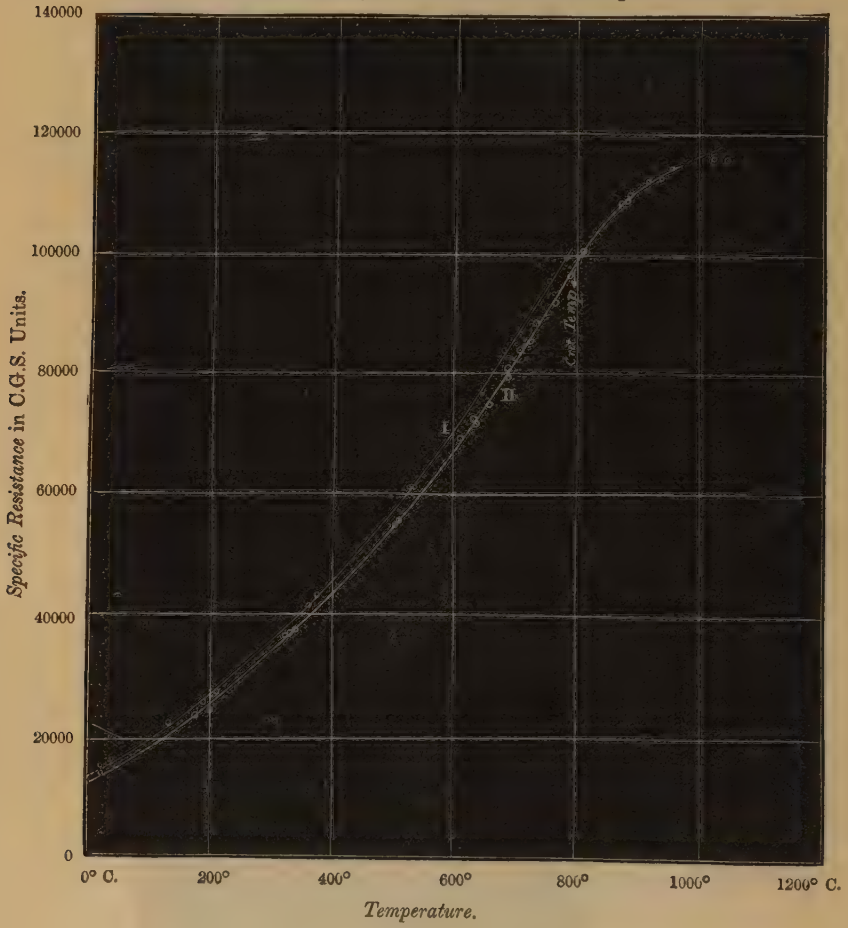


Resistance-Temperature Curves.

17.

Iron specimen A. (Crit. temp. 795° C.)

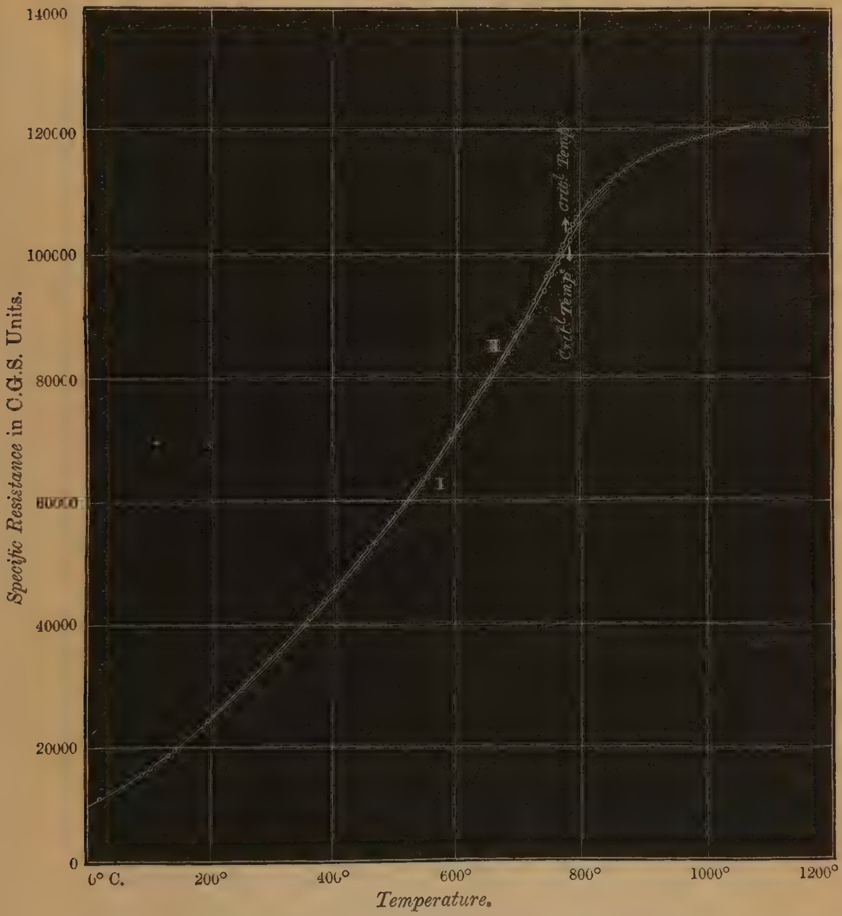
Curve I. Before annealing. Curve II. After annealing at 1050° C.



Resistance-Temperature Curves.

18. Iron specimen B. (Crit. temp. 780° C.)

Curve I. After annealing at 840° C. Curve II. After annealing at 1150° C.



XV. *A new Definition of Focal Length, and an Instrument for determining it.* *By* T. H. BLAKESLEY, *M.A.*[*]

I HAVE been frequently struck with the very imperfect way in which the facts of focussing with lenses and lens-combinations are treated in works which purport to deal with such matters in a practical way. In many cases such works are by authors who in other respects give information and rules which are unimpeachable and the result of painstaking study, so that the defect in the particular to which I have alluded must be due either to the inherent inconvenience in the usual formulæ, so invariably and unnecessarily involving the inverse of distances of Object and Image from theoretically fixed but practically undetermined points, and the Focal Length; or to the imperfect way of quoting the rules in the theoretical text-books. If the latter be even partially the cause, and can be remedied, it will be quite unnecessary to inquire if the former is also operative. In addition to the troublesome formula of inverse distances, a difficulty is unnecessarily introduced by defining the focal length as the distance between two points, generally the principal focus and a focal point, or the lens itself if thin, instead of being an abstract length of straight line characteristic of the lens or lens-combination.

To illustrate my meaning, I may point out that the Coefficient of Self-Induction of a coiled conductor is expressed as a length which is fixed so long as the coil remains unaltered in geometrical conditions; but no one would ask between which points is this length to be measured. So the focal length of a lens-combination is simply an abstract length, and not necessarily the distance between two particular points.

But the Royal Society method of finding the focal length of a lens-system, as carried out at the Kew establishment, is based on the definition that the focal length is the distance between a focal centre and the principal focus; and these points having been separately found by processes not free from objection, the distance between them is indirectly measured

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[*] Read June 11, 1897.

(*vide* Major Darwin's paper, 'Proceedings of the Royal Society,' 1892).

Various methods have been described in an admirably arranged bibliographical review of the subject by Prof. S. P. Thompson. In this paper the author also describes a method of his own with an apparatus for carrying it out, but that method depends on the same idea that the focal length is a distance between two points. The apparatus fixes those points and measures the distance between them. And the same may be said of all but one of the methods proposed by others, described in that paper.

I have to propose and explain a far more general definition of the focal length of any system of Coaxial Lenses, which leads naturally to more general methods of determining that length than any of those alluded to.

Now any function of the two distances of object and image from their appropriate focal centres and the focal length may be employed in conjunction with the usual formula to eliminate either of those distances; and if that function is at the same time an easy one to determine experimentally, and if the elimination leads to a simple formula for working purposes, a considerable gain in convenience and exactitude may be the result. Such a function is found in the magnification, which I define as the linear relation of Image to Object and which I symbolize under the letter m, and take as positive if the image is erect and is not inverted with regard to the object. Thus in the case of the object and image being at equal distances from, and on opposite sides of a thin double-convex lens of equal radii of curvature, I should describe the magnification as -1. In this notation the negative sign will indicate reversal, the numeral the arithmetic magnification.

Let A be the position of one of a pair of conjugate foci on the axis of some coaxial lens-system, and let v be its distance from any fixed point on the axis measured positively in the direction the light is passing. Let m be the magnification, or the linear ratio of an image formed at this point to the object producing it.

Then it is easily proved that $\frac{dv}{dm}=f$ a constant; and this constant is in fact the focal length.

Hence $$v-v_0=f.m,$$

where v_0 is the value of v when the magnification is zero, *i. e.* at the principal focus.

It follows from the constancy of $\frac{dv}{dm}$ that dv and dm need not be infinitesimals; so that if any two positions of A be found for a clear image of an object moved in any way along the axis, the distance between these positions bears to the focal length a ratio which is simply the change in the magnification.

Similarly, if u is the position of the other focus but m retains its first signification,

$$\frac{du}{d\left(\frac{1}{m}\right)} \text{ is constant } = f',$$

and

$$u-u_0=f'.\frac{1}{m},$$

where u_0 is the value of u when m is infinite.

Incidentally,

$$\overline{u-u_0}\,\overline{v-v_0}=f.f',$$

$$\frac{v-v_0}{u-u_0}=\frac{f}{f'}m^2.$$

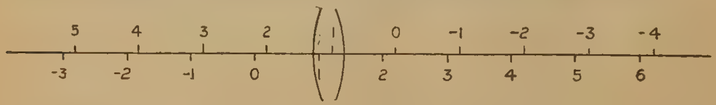
If the end media are alike, $f=f'$, which is the important case

$$\overline{u-u_0}\,\overline{v-v_0}=f^2,$$

$$\frac{v-v_0}{u-u_0}=m^2.$$

The former of these equations is a common one, but the latter is not so well recognized.

m^2 may be called the areal magnification. It clearly has an importance in the question of photographic exposure.



In the sketch let the straight line represent the axis of a lens-system, and let the points marked off at equal distances (f) above the line be numbered in such a way that the number represents the relative size of the Object to the Image, when the Object is placed in any one of the points, *i. e.* for the focus of light before incidence, and let the points numbered below the line, and also at equal distances (f_1), bear a corresponding series of numbers representing the relative size of Image and Object when the Image is at any one of the points.

Then any point represented by p on one of these scales is conjugate to the point represented by $\frac{1}{p}$ on the other. The two principal foci must be marked 0 and the two focal centres must be marked 1.

If the media at the ends are the same, the common distance of the integral points will be the same in the two scales. This condition, as being the common one, I shall assume to exist unless the reverse is especially stated.

The common distance between any two consecutive integral points is the focal length; and this is one important characteristic of the system. Another characteristic is the aberration of the integral points of one scale from those of the other. A third is the order in which the numbers of the scales run with reference to the direction of the light.

It appears to me that two systems to deserve being called equivalent one to the other should have coincidence in all these three respects. If the top series has reference to light before encounter with the system, and light is supposed to come from right to left, then for a convex arrangement the upper series of numbers increases in the same direction, *i. e.* from right to left, and the lower series in the reverse direction; but for a concave arrangement these rules are reversed.

It will be found that one of these diagrams will not be misinterpreted by turning it upside down.

Hence in any combination it is necessary to find:—

(1) The focal length.
(2) Some point of known numerical denomination on one scale.
(3) Ditto for the other scale.

With these three determinations all the facts connected with image-formation as regards position and size may be determined for any position of the object, and *vice versâ*.

The optical bank with which the focal length is determined contains various sliding-pieces in a mahogany groove. Two of them carry pieces for the support of the cells containing the lens-combination, and two carry frames in which can be placed photographs on lantern-plate glasses of evenly-divided scales.

These photographs, being printed off the same negative, are exactly alike, and in use are turned with the film-side inwards towards the lenses.

If these photographs are made to occupy the positions of object and image, a comparison of the divisions actually on the plate examined with those of the image of the other scale received upon it will readily give the magnification.

Groups of exactly ten divisions, at ten divisions' interval, are also indicated by dark bars for the convenience of reading. An anatomical magnifier standing behind the scale serves to adjust to coincidence the planes of the actual and the image-scale with great accuracy; and it is also obvious that for any one position of the sliding-pieces the magnifier may be applied at both ends successively, each observation being thus checked by another.

To measure the movement of any sliding-piece a brass ruler is screwed to it at right angles to the motion; a slip of telegraph exchange-paper is fixed by drawing-pins to cork pieces at either end of the bank and the rulers pass immediately over this slip, so that lines can be drawn on the paper by a drawing-pen or pencil at any position, and the range of motion accurately measured.

To employ the instrument for convex lens-combinations, such as photographic lenses and projectors, several methods can be adopted. The one scale and the combination being fixed, search may be made for the image, which is adjusted accurately to the other scale, and a mark made on the paper at the edge at the corresponding ruler, and the magnification determined and checked, if necessary, at the other end.

The first scale may be then moved any distance, and the new position of the image found and adjusted as before by moving the second scale. A new mark is made, and the new magnification found and checked.

If m and n are the two magnifications and l the distance between the marks,

$$\frac{l}{m-n}=f.$$

Or the two scales may remain fixed and the combination may be moved until accurate adjustment is obtained in the two possible positions; the distance between those two positions being obtained by marks made on the paper along the edge of the ruler attached to the combination.

In this case the magnification which is the inverse in one of the positions to what it is in the other may be ascertained by *four* readings. Then obviously $n=\frac{1}{m}$ and $f=\frac{l}{m-\frac{1}{m}}$ $=\frac{l\,.\,m}{\overline{m+1}\cdot\overline{m-1}}$, a very convenient form.

The slips of paper employed may be rolled up and stored, so that reference can be made to them should occasion require it.

The operation of finding the focal length of a concave lens or combination is less important perhaps, but the following general proposition may be of interest, and within limits may be made applicable to any case.

Suppose it possible to change the position of a portion of a combination, so as to recover as in the last operation the same pair of conjugate foci of that part of the combination. Then clearly the focussing of the whole combination will be restored to the same conjugate foci as before, though the magnification will be changed.

Let m be the factor of the total magnification due to the moving portion, and let M be the total magnification: in the first position M may be read off.

In the second position m changes to $\frac{1}{m}$, and M changes to N, say, which may also be determined.

If K be the magnification-factor due to the stationary portion of the combination,

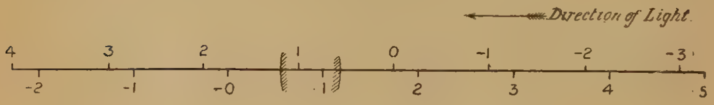
$$\mathrm{K}m=\mathrm{M}, \qquad \frac{\mathrm{K}}{m}=\mathrm{N};$$

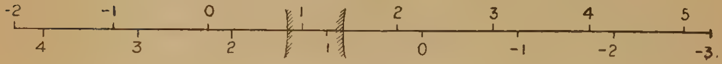
from which m and K can both be determined, viz.

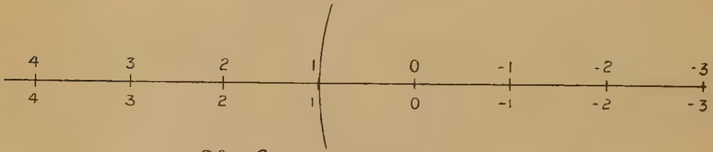
$$m^2=\frac{\mathrm{M}}{\mathrm{N}}, \qquad \mathrm{K}^2=\mathrm{MN}.$$

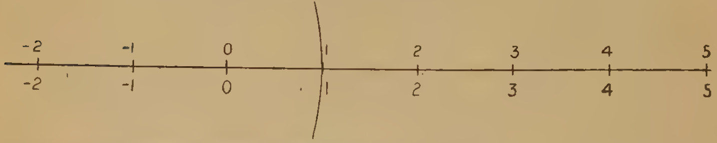
Hence the focal length of the moving part is

$$\frac{lm}{m^2-1} \quad \text{or} \quad \frac{l\sqrt{MN}}{M-N}.$$


TYPE OF A CONVEX SYSTEM


D° CONCAVE SYSTEM


D° CONCAVE MIRROR


D° CONVEX MIRROR

Note.—In all these diagrams light is supposed to come in *from the right hand*; the upper scale applies to the focus for the light before it encounters the system, the under scale to the focus for the light which has emerged from the system. The diagrams hold good if the sheet is turned upside down. Any erecting system of lenses applied to a given combination changes the character of the latter from concave to convex and *vice versâ.*

I have dealt with the finding of focal lengths before showing how any particular focus may be found, with a view of accentuating how completely independent the two matters are; but it is clear that if any position of a pair of conjugate foci is taken and the magnification is measured, one has only to measure the distance of the scales from some fixed planes of the lens-system (say the edges of the brass cells) in order to allocate not merely the points for the particular magnification, but for all magnifications both above and below the line.

Such measurements could easily be made by a sliding rule, accurately to the tenth part of a millimetre; but as the end of the rule would of necessity abut against the photographic film, repeated measurements would be prejudicial to the latter. However, if the negative from which the scales are made is preserved, such scales can be easily and cheaply reproduced.

A very accurate way of fixing the principal foci is to place the combination at rest vertically on a plane mirror. Immediately above it hold a sharply-cut point of stiff white paper in a universal holder and adjust its position in the vertical line until its image formed by light passed twice through the lens-system coincides with itself. This adjustment may be most accurately made by a short focal-length hand-lens.

DISCUSSION.

Dr. S. P. THOMPSON said the paper was the most important contribution to geometrical optics that had appeared for many years. The introduction of the magnification function was a most useful device, leading to exceedingly simple results. The important thing to measure was not so much the focal length as the reciprocal of that quantity.

Dr. CHREE pointed out that in employing the Kew method it was not of great importance that the axis round which the lens-system is moved should pass through the nodal point. A small error in this adjustment imported none into the final result. He wished to refer to Hondaille's book on Lens Testing. A magnification method is given in that work.

Mr. BLAKESLEY, in replying, called attention to the use of his strip diagrams of magnification, for enlarging purposes in photography. When the magnification diagram was known, the focussing-cloth might be dispensed with.

XVI. *Note on* Mr. Blakesley's *paper, "A new Definition of Focal Length, &c."* *By* Prof. A. GRAY*.

I HAVE read Mr. Blakesley's paper on "A new Definition of Focal Length, &c.," with much interest. It is true, as Mr. Blakesley states, that the treatment of lenses and combinations of lenses in ordinary text-books is frequently faulty, and that much advantage would be gained by a consideration of the question from other points of view than that generally adopted in treatises on geometrical optics. Something of this kind has been achieved by the admirable series of papers on points in physical optics which have appeared from time to time, during the last fifteen years, from the pen of Lord Rayleigh, and the lucid and elementary discussion of the propagation of waves through lenses, and their reflexion from mirrors, published by Dr. S. P. Thompson, in Oct. 1889.

I think further clearness in the presentment of the action of lens systems in important cases would be obtained if more use were made of the notion of the apparent distance of an object seen through a system of lenses. This idea, which is at least as old as Smith's 'Compleat System of Opticks' (Camb. 1738), seems to have been strangely neglected until attention was called to it again by Lord Rayleigh in the Phil. Mag. for June 1886. Yet the formula for the apparent distance of an object situated on or near the common axis of a system of lenses, and viewed along that line, yields at once from its mere form many most valuable theorems: for example, that the interchange of position of the image and object, without change of position of the lens-system, does not affect the magnification; that the magnification of an object, seen through such a lens-system, is equal to the ratio of the real distance of the object from the object-glass to the apparent distance of the object from the eye, or the ratio of the breadth of the pencil at the object-glass to the breadth of the pencil at the eye.

In this mode of discussion attention is, as in Mr. Blakesley's paper, focussed on the magnification produced by the lens-system. Hence the method of determining the focal length of a combination by comparisons of the magnifications for (1) two positions of the object at a measured distance apart, or for (2) two positions of the image at a measured distance apart. If l denote the distance in either case, m, n the magnifications, the focal length is $l\frac{mn}{m-n}$ in (1), and $\frac{l}{m-n}$ in the other; or if, as Mr. Blakesley suggests, the two possible positions of the combination for fixed positions of the object and image be taken, and l be the distance through which the combination is moved, the focal length is $\frac{lm}{m^2-1}$.

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* Read June 11, 1897.

But this method of determining focal lengths has already been used for several years by Abbe for his optical combinations, and apparatus for the purpose is to be found described by S. Czapski (*Zeitschrift für Instrumentenkunde*, vol. xii. 1892). A discussion of the method and some account of the apparatus will be found in the very valuable treatise on "Optik," by Czapski, lately published as part of the *Handbuch der Physik*, just completed under the editorship of Dr. A. Winkelmann (see vol. ii. pp. 289, 290, *et seq.*).

Though the experimental method of Mr. Blakesley's paper has thus been in the main anticipated, his paper is valuable as a fresh and instructive view of the subject, and as directing attention to methods of focometry as yet apparently not very generally known.

Bangor, June 17, 1897

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XVII. *The Effect of a Spherical Conducting Shell on the Induction at a Point in the Dielectric outside due to an Alternating Current in a Circular Circuit in the Dielectric inside, the Axis of the Conductor passing through the Centre of the Shell.* *By* C. S. WHITEHEAD, *M.A.**

(1) IT is well known that if there be an alternating current in a primary circuit sounds will be heard in a telephone in a secondary circuit, even if the latter be at a considerable distance from the former. If the sounds are preconcerted, signals may be, and in fact have been, transmitted in this manner on land through distances of four or five miles.

It has been suggested that the same plan could be employed to communicate with lightships. The primary is laid on the sea-bottom round the area over which the lightship swings, the ends of the cable being brought to land; the secondary is coiled round the lightship. I have been informed by Mr. S. Evershed that successful experiments were carried out by these means last summer in Dover harbour, the depth of the sea being about 1000 centims. To be able to judge whether a similar experiment will succeed in a greater depth of water, we must calculate the induction through the secondary circuit; sea-water being a conductor the investigation is somewhat complex, the final result, however, comes out in a simple form.

(2) The two following cases are considered :—

CASE I.

A circular circuit carrying an alternating current is placed in the dielectric inside a spherical conducting shell, the axis of the circuit passing through the centre of the shell: to find the normal magnetic induction at any point in the dielectric outside the shell.

CASE II.

The circular circuit is placed on one side of an infinite conducting plate, the plane of the circuit being parallel to the plate: to find the normal magnetic induction at any point on the other side of the plate.

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* Read June 11, 1897.

In both cases the following result is arrived at:—

$$\frac{v_0}{u_0}=e^{-q\eta},$$

where v_0 is the maximum value of the normal magnetic induction at any point outside, u_0 the maximum value of the normal magnetic induction due to the current in the primary, supposing the conducting shell or plate absent, at the same point, η is the thickness of the shell or plate,

$$q=\left(\frac{2\pi\mu p}{\sigma}\right)^{\frac{1}{2}},$$

μ is the permeability of the conducting shell or plate, σ its specific resistance, $p=2\pi$ times the frequency.

Let the frequency be 300, which makes $p=1885$; σ for sea-water $=2.10^{10}$ C.G.S. units, $\mu=1$.

If $\eta=2000$ centims., which is about the depth of the sea by the North Sand Head Lightship off the Goodwin Sands, then

$$\frac{v_0}{u_0}=\cdot 21, \quad \text{and 79 per cent. is lost.}$$

If

$$\eta=1000, \quad p=1885,$$

$$\frac{v_0}{u_0}=\cdot 46, \quad \text{and 54 per cent. is lost.}$$

The above value of σ is from experiments on a sample of sea-water from the North Sea made by Mr. S. Evershed.

(3) Case I.

Take O the centre of the shell for origin, and axis of the circuit for axis of z.

Let D be the centre of the circuit, C any point in it.

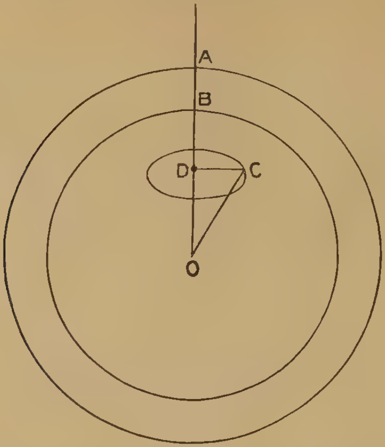
OA = external radius of the shell $=a$.

OB = internal radius of the shell $=b$.

OC $=c$, $\angle$ DOC $=\alpha$.

$a-b=\eta$.

Let P, Q, R be the components of the electromotive intensity parallel to the axes of x, y, z respectively, σ the specific resistance of the shell, μ the permeability of the shell, μ' that of the dielectric, K′ the specific inductive capacity of the dielectric, r, θ, ϕ the polar coordinates of any point, θ being the colatitude, ϕ the longitude.



Let the current in the circuit be represented by the real part of $\gamma e^{\iota pt}$, $p=2\pi\times$ frequency, $\iota=(-1)^{\frac{1}{2}}$.

By symmetry the induced currents in the shell will flow in circles parallel to the plane of the circuit.

Let Φ be the electromotive intensity tangential to one of these circles;

$$\therefore\ \mathrm{P}=-\sin\phi\,.\,\Phi,\quad \mathrm{Q}=\cos\phi\,.\,\Phi,\quad \mathrm{R}=0.\ .\ .\ .\ .\ . \qquad (1)$$

Φ is by symmetry independent of ϕ.

Now P satisfies the equation

$$\nabla^2\mathrm{P}=\frac{4\pi\mu}{\sigma}\frac{d\mathrm{P}}{dt}\ \text{in the conductor,}$$

and

$$\nabla^2\mathrm{P}=\mu'\mathrm{K}\frac{d^2\mathrm{P}}{dt^2}\ \text{in the dielectric.}$$

Assume

$$\mathrm{P}\ \text{varies as}\ e^{\iota pt},$$

and let

$$\lambda^2=-\frac{4\pi\mu\iota p}{\sigma},$$

$$\lambda'^2=\ \mu'\mathrm{K}'p^2.$$

$$\therefore\qquad \nabla^2\mathrm{P}+\lambda^2\mathrm{P}=0\,;\ .\ .\ .\ .\ .\ .\ .\ . \qquad (2)$$

in the dielectric we must write λ' for λ.

Q and R satisfy equations of the same form.

Assume

$$\left.\begin{aligned}P&=\psi_n(\lambda r)\left(y\frac{d}{dz}-z\frac{d}{dy}\right)\chi_n\,e^{\iota pt}\\ Q&=\psi_n(\lambda r)\left(z\frac{d}{dx}-x\frac{d}{dz}\right)\chi_n\,e^{\iota pt}\\ R&=\psi_n(\lambda r)\left(x\frac{d}{dy}-y\frac{d}{dx}\right)\chi_n\,e^{\iota pt}\end{aligned}\right\}\quad .\ .\ .\ .\ (3)$$

These equations make $\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}=0$, as should be the case.

χ_n is an arbitrary solid spherical harmonic of degree n.

Substituting in (2) we find

$$\frac{d^2\psi_n}{dr^2}+\frac{2(n+1)}{r}\frac{d\psi_n}{dr}+\lambda^2\psi_n=0.\quad .\ .\ .\ (4)$$

Since $R=0$, the last equation of (3) gives

$$\frac{d\chi_n}{d\phi}=0.$$

∴ from the first of (3).

$$P=-\sin\phi\,.\,\psi_n(\lambda r)\frac{d\chi_n}{d\theta}e^{\iota pt}.$$

$$\therefore\ \Phi=\psi_n(\lambda r)\frac{d\chi^n}{d\theta}e^{\iota pt}.\quad .\ .\ .\ .\ .\ .\ (5)$$

Let ω = magnetic induction tangential to a meridian,
v = magnetic induction along a radius.

Then from the theorem that the line integral of the electromotive intensity round a circuit is equal to the rate of decrease of the magnetic induction through the circuit, we obtain

$$\frac{d\omega}{dt}=-\frac{1}{r}\frac{d(\Phi r)}{dr},$$

$$\frac{dv}{dt}=\frac{1}{r\sin\theta}\frac{d(\Phi\sin\theta)}{d\theta}.$$

But Φ varies as $e^{\iota pt}$,

$$\therefore\ \omega=-\frac{1}{\iota pr}\frac{d(\Phi r)}{dr},$$

$$v=\frac{1}{\iota pr}\frac{1}{\sin\theta}\frac{d(\Phi\sin\theta)}{d\theta}.$$

Since $\frac{d\chi_n}{d\phi}=0$, χ_n satisfies

$$\frac{d}{dr}\left(r^2\frac{d\chi_n}{dr}\right)+\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\chi_n}{d\theta}\right)=0,$$

∴ from (5)

$$\omega=-\frac{1}{\iota pr}\frac{d^2}{d\theta . dr}\left\{r\psi_n(\lambda r)\chi_n\right\}e^{\iota pt},$$

$$v=-\frac{1}{\iota pr}\psi_n(\lambda r)\frac{d}{dr}\left(r^2\frac{d\chi_n}{dr}\right)e^{\iota pt}.$$

But

$$r\frac{d\chi_n}{dr}=n\chi_n,$$

$$\therefore\ \frac{d}{dr}\left(r^2\frac{d\chi_n}{dr}\right)=n(n+1)\chi_n,$$

and

$$\frac{d}{dr}\left\{r\psi_n(\lambda r)\chi_n\right\}$$

$$=\left\{\frac{d . r\psi_n(\lambda r)}{dr}+n\psi_n(\lambda r)\right\}\chi_n,$$

$$=\phi_n(\lambda r)\chi_n,$$

where

$$\phi_n(\lambda r)=\frac{d . r\psi_n(\lambda r)}{dr}+n\psi_n(\lambda r), \quad . \quad (6)$$

$$\left.\begin{aligned}\therefore\ \omega&=-\frac{1}{\iota pr}\phi_n(\lambda r)\frac{d\chi_n}{d\theta}e^{\iota pt}\\ v&=-\frac{n(n+1)}{\iota pr}\psi_n(\lambda r)\chi_n e^{\iota pt}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (7)$$

Again, μ' K$'$ is very small (it is of the order 10^{-21}), hence, unless p is very large, we may neglect λ'^2. Hence in the dielectric

$$\nabla^2\mathrm{P}=0,$$

we may therefore assume

$$\mathrm{P}=\left(y\frac{d}{dz}-z\frac{d}{dy}\right)\mathrm{X}^n e^{\iota pt},$$

where X_n is an arbitrary solid spherical harmonic, so that in the dielectric we must put $\psi_n(\lambda' r)=1$, and, consequently, $\phi_n(\lambda' r)=n+1$.

Therefore in the dielectric

$$\left.\begin{aligned} v &= -\frac{n(n+1)}{\iota pr}\,X_n\,e^{\iota pt} \\ \omega &= -\frac{n+1}{\iota pr}\,\frac{dX_n}{d\theta}\,e^{\iota pt} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The inducing system is in the dielectric inside the shell. Hence in the dielectric inside the shell

$$\left.\begin{aligned} v &= -\frac{n(n+1)}{\iota pr}\,(X_n+X_{-n-1})e^{\iota pt} \\ \omega &= -\frac{1}{\iota pr}\left\{(n+1)\frac{dX_n}{d\theta}-n\,\frac{dX_{-n-1}}{d\theta}\right\}e^{\iota pt} \end{aligned}\right\} \quad . \quad . \quad . \quad (9)$$

In the dielectric outside the shell

$$\left.\begin{aligned} v &= -\frac{n(n+1)}{\iota pr}\,Z_{-n-1}\,e^{\iota pt} \\ \omega &= \frac{n}{\iota pr}\,\frac{dZ_{-n-1}}{d\theta}\,e^{\iota pt} \end{aligned}\right\}, \quad . \quad . \quad . \quad (10)$$

where Z_{-n-1} is an arbitrary solid spherical harmonic of degree $-n-1$.

In the conducting shell

$$\left.\begin{aligned} v &= -\frac{n(n+1)}{\iota pr}\,\psi_n(\lambda r)\chi_n\,e^{\iota pt} \\ \omega &= -\frac{1}{\iota pr}\,\phi_n(\lambda r)\,\frac{d\chi_n}{d\theta}\,e^{\iota pt} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (11)$$

The boundary conditions are (1) that the normal magnetic induction, (2) that the tangential magnetic force is to be continuous. The tangential magnetic force $=\frac{\omega}{\mu}$ in the conductor, and $=\omega$ in the dielectric.

$$\left.\begin{aligned} \therefore\ \psi_n(\lambda b)\chi_n &= X_n+X_{-n-1} \\ \psi_n(\lambda a)\chi_n &= Z_{-n-1} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (12)$$

$$\left.\begin{aligned} \frac{1}{\mu}\,\phi_n(\lambda b)\,\frac{d\chi_n}{d\theta} &= (n+1)\frac{dX_n}{d\theta}-n\,\frac{dX_{-n-1}}{d\theta} \\ \frac{1}{\mu}\,\phi_n(\lambda a)\,\frac{d\chi_n}{d\theta} &= -n\frac{dZ_{-n-1}}{d\theta} \end{aligned}\right\} \quad . \quad . \quad (13)$$

From these equations we obtain

$$\frac{X_n + X_{-n-1}}{Z_{-n-1}} = \frac{b^n \psi_n(\lambda b)}{a^n \psi_n(\lambda a)},$$

$$\frac{(n+1)X_n - nX_{-n-1}}{-nZ_{-n-1}} = \frac{b^n \phi_n(\lambda b)}{a^n \phi_n(\lambda a)};$$

$$\therefore\ b^n\{(n+1)\phi_n(\lambda a)\psi_n(\lambda b) + n\phi_n(\lambda b)\psi_n(\lambda a)\}Z_{-n-1}$$
$$= a^n(2n+1)\phi_n(\lambda a)\psi_n(\lambda a)X_{-n-1}. \quad . \quad (14)$$

Let λa and λb be so large that $\frac{1}{\lambda a}$, $\frac{1}{\lambda b}$ may be neglected

$\psi_n(\lambda r)$ satisfies

$$\frac{d^2\psi_n}{dr^2} + \frac{2(n+1)}{r}\frac{d\psi_n}{dr} + \lambda^2\psi_n = 0.$$

Assume

$$\psi_n = \frac{e^{mr}}{r^{n+1}}u,$$

and let

$$\lambda^2 + m^2 = 0,$$

$$\therefore\ \frac{d^2u}{dr^2} + 2m\frac{du}{dr} - \frac{n(n+1)}{r^2}u = 0.$$

Let

$$u = a_0 + \frac{a_1}{r} + \frac{a_2}{r^2} + \ldots$$

Substituting, we find in the usual manner,

$$u = 1 - \frac{n(n+1)}{2mr} + \frac{(n-1)n(n+1)(n+2)}{1 \,.\, 2 \,.\, (2mr)^2} - \ldots$$

where

$$m = \pm\iota\lambda.$$

Hence, when $\frac{1}{\lambda r}$ is small,

$$\psi_n(\lambda r) = \frac{e^{\iota\lambda r}}{r^{n+1}} \text{ or } \frac{e^{-\iota\lambda r}}{r^{n+1}}.$$

Now Φ is zero at infinity and must decrease as r increases,

$$\therefore\ \psi_n(\lambda r) = \frac{e^{-\iota\lambda r}}{r^{n+1}},$$

$$\therefore\ \phi_n(\lambda r) = -\iota\lambda\frac{e^{-\iota\lambda r}}{r^n}.$$

Substituting in (14)

$$Z_{-n-1}=\frac{(2n+1)be^{-\iota\lambda\eta}}{(n+1)a+nb}\,X_{-n-1}$$

$$=e^{-\iota\lambda\eta}\,X_{-n-1}\quad .\quad .\quad .\quad .\quad .\quad .\quad .\qquad (15)$$

if both a and b be large compared with their difference.

Let Ω denote the solid angle subtended by the circuit at any point;

V the potential due to the circuit;

U the normal magnetic force.

$$\Omega=2\pi\sin^2\alpha\sum_1^\infty\frac{1}{n+1}\left(\frac{c}{r}\right)^{n+1}P_n'(\alpha)P_n(\theta),$$

if $r>c$. P_n is the nth zonal harmonic.

$$V=\gamma e^{\iota pt}\Omega,$$

$$U=-\frac{dV}{dr},$$

$$=2\pi\gamma\sin^2\alpha e^{\iota pt}\frac{1}{r}\sum_1^\infty\left(\frac{c}{r}\right)^{n+1}P_n'(\alpha)P_n(\theta).\quad .\quad .\quad .\quad .\qquad (16)$$

Now in the dielectric inside the shell,

$$v=-\frac{n(n+1)}{\iota pr}(X_n+X_{-n-1})e^{\iota pt},$$

$$\therefore\ -\frac{n(n+1)}{\iota p}X_{-n-1}=2\pi\gamma\sin^2\alpha\left(\frac{c}{r}\right)^{n+1}P_n'(\alpha)P_n(\theta),$$

$$\therefore\ X_{-n-1}=-\frac{2\pi\gamma\sin^2\alpha\,\iota p}{n(n+1)}\left(\frac{c}{r}\right)^{n+1}P_n'(\alpha)P_n(\theta),\quad .\quad .\qquad (17)$$

$\therefore$ from (15)

$$Z_{-n-1}=-e^{-\iota\lambda\eta}\frac{2\pi\gamma\sin^2\alpha\,.\,\iota p}{n(n+1)}\left(\frac{c}{r}\right)^{n+1}P_n'(\alpha)P_n(\theta).\quad .\qquad (18)$$

But in the dielectric outside the shell

$$v=-\frac{n(n+1)}{\iota pr}Z_{-n-1}e^{\iota pt},$$

$$\therefore\ v=e^{-\iota\lambda\eta}2\pi\gamma\sin^2\alpha\,e^{\iota pt}\frac{1}{r}\left(\frac{c}{r}\right)^{n+1}P_n'(\alpha)P_n(\theta).$$

Now $$\lambda^2=-\frac{4\pi\mu\iota p}{\sigma}=(1-\iota)^2q^2,$$

where

$$q^2 = \frac{2\pi\mu p}{\sigma},$$

$$\therefore \text{ real part of } e^{-\iota\lambda\eta} e^{\iota pt} = \text{real part of } e^{-q\eta} e^{\iota(pt-q\eta)}$$

$$= e^{-q\eta} \cos(pt - q\eta),$$

$$\therefore\ v = e^{-q\eta} 2\pi\gamma \sin^2\alpha \frac{1}{r}\left(\frac{c}{r}\right)^{n+1} \mathrm{P}_n'(\alpha)\mathrm{P}_n(\theta)\cos(pt - q\eta). \quad . \quad (19)$$

Let v_0 be the maximum value of v, U_0 that of U ;

∴ from (16) and (19)

$$\frac{v_0}{\mathrm{U}_0} = e^{-q\eta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

The method employed in this investigation is taken from a paper by Professor H. Lamb, Phil. Trans. Part ii. 1883.

(4) Case II.

The result for this case may be deduced from the preceding by using a particular case of a transformation due to Professor C. Niven (Phil. Trans. Part ii. 1883).

He shows that if P_n denote a zonal harmonic of the nth degree, $s = \sin\theta$, $n = ka$, $s = \frac{\rho}{a}$; then when n and a become infinite, k and ρ remaining finite,

$$\mathrm{P}_n = \mathrm{J}_0(k\rho),$$

$$\mathrm{P}_n' = \frac{ka^2}{\rho}\mathrm{J}_1(k\rho),$$

J_0 and J_1 being Bessel functions.

To find the value of Ω in terms of Bessel's functions. Let P be any point, draw PM perpendicular to the axis of z.

Let CD, the radius of the circuit, $= f$,

$\mathrm{PM} = \rho$, $\angle\mathrm{POM} = \theta$, $\angle\mathrm{COD} = \alpha$, $\mathrm{OC} = c$, $\mathrm{DM} = z$, $\mathrm{OP} = r$.

$$\Omega = 2\pi \sin^2\alpha \sum_1^\infty \frac{1}{n+1}\left(\frac{c}{r}\right)^{n+1} \mathrm{P}_n'(\alpha) \,.\, \mathrm{P}_n(\theta), \quad r > c.$$

Let

$$n = kr = k_1 c,$$

$$\sin\theta = \frac{\rho}{r}, \quad \sin\alpha = \frac{f}{c}.$$

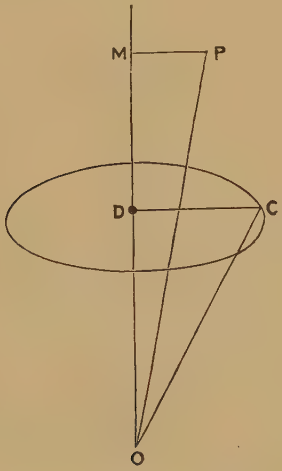
Let n, r, c become infinite, k, ρ, f remaining finite;
$\therefore$ ultimately $k=k_1$,

$$\mathrm{P}_n(\theta)=\mathrm{J}_0(k\rho),\quad \mathrm{P}_n'(\alpha)=\frac{kc^2}{f}\mathrm{J}_1(kf),$$

$$c=\mathrm{OD}=\mathrm{OM}-\mathrm{DM}$$

$$=r-z \text{ ultimately};$$

$$\therefore \left(\frac{c}{r}\right)^n=\left(1-\frac{z}{r}\right)^{kr}=e^{-kz} \text{ in the limit.}$$



The successive values of n are 1, 2, 3...; let $k+dk$ be the successive values of k;

$$\therefore n+1=(k+dk)r;$$

$$\therefore rdk=1;$$

$$\therefore \frac{k}{n+1}\left(\frac{c}{r}\right)=\frac{n}{n+1}dk=dk;$$

$$\therefore \Omega=2\pi f\int_0^\infty e^{-kz}\mathrm{J}_0(k\rho)\mathrm{J}_1(kf)dk.$$

This result can also be deduced from the equation

$$\frac{d^2\Omega}{dr^2}+\frac{1}{r}\frac{d\Omega}{dr}+\frac{d^2\Omega}{dz^2}=0,$$

remembering that

$$\text{when } z=0,\ \Omega=2\pi \text{ from } r=0 \text{ to } r=a.$$
$$\Omega=0 \quad \text{from } r=a \text{ to } r=\infty.$$

We thus find

$$v=2\pi\gamma e^{-q\eta} f\cos(pt-q\eta)\int_0^\infty ke^{-kz}J_0(k\rho)J_1(kf)dk,$$

$$U=2\pi\gamma f\cos pt\int_0^\infty ke^{-kz}J_0(k\rho)J_1(kf)dk;$$

$$\therefore\ \frac{v_0}{U_0}=e^{-q\eta}$$

as before.

APPENDIX.

It may be useful to add the proof of the transformation used in Case II.

If $\mu=\cos\theta$,

P_n satisfies

$$(1-\mu^2)\frac{d^2P_n}{d\mu^2}-2\mu\frac{dP_n}{d\mu}+n(n+1)P_n=0. \quad . \quad . \quad (1)$$

Let $s=\sin\theta$.

(1) transforms into

$$(1-s^2)\frac{d^2P_n}{ds^2}+\frac{1-2s^2}{s}\frac{dP_n}{ds}+n(n+1)P_n=0.$$

Assuming

$$P_n=a_0+a_1s+a_2s^2+\ldots$$

we find in the usual manner

$$P_n=a_0\left\{1-\frac{n(n+1)}{2^2}s^2+\frac{(n-2)n(n+1)(n+3)}{2^2.4^2}s^4-\ldots\right\}.$$

But $P_n=1$ when $s=0$;

$$\therefore\ a_0=1.$$

Let $n=ka,\quad s=\dfrac{\rho}{a},$

and let n and a become infinite, k and ρ remaining finite.

$$\frac{n(n+1)}{2^2}s^2=\frac{k^2\rho^2}{2^2},$$

ultimately

$$\frac{(n-2)n(n+1)(n+3)}{2^2 \cdot 4^2}s^4 = \frac{k^4\rho^4}{2^2 \cdot 4^2},$$

$$\&c. \qquad = \qquad \&c.$$

∴ ultimately

$$P_n = 1 - \frac{k^2\rho^2}{2^2} + \frac{k^4\rho^4}{2^2 \cdot 4^2} - \frac{k^6\rho^6}{2^2 \cdot 4^2 \cdot 6^2} + \dots$$

$$= J_0(k\rho).$$

Again

$$s^2 + \mu^2 = 1;$$

$$\therefore\ sds = -\mu d\mu,$$

$$\mu^2 = 1 - s^2 = 1 \text{ ultimately},$$

$$s = \frac{\rho}{a}; \quad \therefore\ ds = \frac{d\rho}{a};$$

$$\therefore\ d\mu = -s\frac{d\rho}{a} = -\frac{\rho d\rho}{a^2};$$

$$\therefore\ P_n' = \frac{dP_n}{d\mu} = -\frac{a^2}{\rho}\frac{dJ_0(k\rho)}{d\rho}$$

$$= \frac{ka^2}{\rho}J_1(k\rho).$$

DISCUSSION.

Mr. EVERSHED referred to some experiments of his own, from which he concluded that the author's formula gave too low an estimate of the attenuation; the discrepancy indicated that some term had been neglected.

Mr. YULE doubted whether the equations given by the author were quite applicable to sea-water. There was need, apparently, of a term involving the polarisation of the medium.

Mr. HEAVISIDE communicated a criticism of the paper. It was not necessary to investigate the problem for any particular form of circuit from which the waves proceed. The attenuating factor for plane waves, due to Maxwell, was sufficient. Taking the best-known value for the conductivity of sea-water, there was no reason why the conductivity should interfere with signalling. A considerably greater conductivity must be proved for sea-water before it could be accepted that the failure of experiments on telegraphic communication with light-ships from the sea-bottom was due to that factor. It was unlikely theoretically, and Mr. Stevenson had contradicted it from a practical standpoint. For some reason, the account of the light-ship experiments had not been published, so that there was no means of finding the real cause of failure.

XVIII. *On the Decomposition of Silver Salts by Pressure.*

By J. E. MYERS, *M.Sc., Ph.D., late* 1851 *Exhibition Science Scholar, and* F. BRAUN, *Ph.D., Professor of Physics in the University of Strassburg i.E.*[*]

As was demonstrated by Carey Lea[†] some years ago, silver salts and others may be decomposed by the application of pressure. By pounding a portion of the salt in a mortar one may readily effect decomposition.

A convincing proof is obtained by compressing a halogen salt of silver in the absence of sunlight and then subjecting the compressed mass to the developing and fixing processes of photography. A black residue of finely divided silver is the result. Exposure of the salt to sunlight, before compression, makes of course this test useless: the difficulty may be overcome by employing a mixture of $AgNO_3$ and KR ($R=Cl$, Br, or I) in the first instance.

The decomposition is most marked in the case of bromide of silver.

Two different metals immersed in liquid bromine constitute a galvanic element[‡]. The couple Ag, Br, Pt has an E.M.F. of about 0·95 volt. It appeared therefore of interest to determine the E.M.F. developed when AgBr is compressed between electrodes of platinum and silver.

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[*] Read June 11, 1897.

[†] Phil. Mag. xxxi. p. 323 (1891); xxxiv. p. 46 (1892); xxxvi. p. 351. (1893); xxxvii. pp. 31 and 470 (1894).

[‡] F. Braun, Wied. *Ann.* xvii. p. 610 (1862).

The salt was contained in a so-called diamond mortar, which was placed under a press and subjected to a pressure of 3000–5000 atmospheres. The apparatus did not admit of a direct measurement of pressure, but for purposes of comparison, by using the same quantity of salt and turning the lever to the same position the same pressure (approximately) could be applied. The pestle was insulated from the mortar by means of a thin sheet of mica bent into a cylindrical form. A piece of platinum foil beneath the salt, and a short cylinder of silver immediately beneath the pestle, formed the electrodes, which were connected to the quadrants of an electrometer.

The following serves to give a general idea of the run of an experiment :—

(One Daniell element gave a deflexion of 132·1 scale-divisions.)

Ag Br.	Scale-divisions.
On applying pressure, immediate deflexion	1·2
after 2 minutes	2·8
,, 10 ,,	6·8
,, 15 ,,	7·2
,, 40 ,,	9·2
,, 55 ,,	9·0
,, 85 ,,	8·7
,, 95 ,,	8·2
,, 105 ,,	7·0
Pressure removed	5·0

after which the deflexion rapidly sank to zero.

Thus the maximum E.M.F generated was about 0·07 Daniell. The direction of current was from Ag to Pt through the compressed substance.

A higher applied pressure produced an E.M.F. of 0·09 Daniell.

The silver bromide was prepared from pure $AgNO_3$ by precipitation with pure KBr. Before each experiment the salt was carefully dried by heating for two hours or more at a temperature of 150° C., and then allowed to cool in a desiccator. Pure $AgNO_3$ showed no appreciable E.M.F. on compression, neither did KBr: a mixture of the two in equivalent quantities compressed between electrodes of Pt and Ag had an E.M.F. of 0·08 Daniell.

With AgCl and AgI the E.M.F.'s were smaller.

Ag, AgCl, Pt = 0·03 Daniell.
Ag, AgI, Pt = 0·04 „

In both cases a marked darkening in colour was noticeable.

Somewhat remarkable is the fact that under high pressure AgBr becomes gradually transparent even when 3–4 millim. thick. At the same time the electrical resistance increases considerably. For instance, in one case the resistance increased gradually from $3{\cdot}65 \,.\, 10^3$ ohms to $14{\cdot}62 \,.\, 10^3$ ohms (measured with alternating current and telephone) in the course of 48 hours, pressure being continuously applied.

Strassburg, Physik. Institut.

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XIX. *A Method of determining Magnetic Hysteresis loss in Straight Iron Strips.*

By J. A. Fleming, *M.A., D.Sc., F.R.S., Professor of Electrical Engineering in University College, London*[*].

[Plate VII.]

The exact determination of the magnetic hysteresis loss in sheet iron as used for alternating current transformer construction has become of commercial importance of late years. Owing to the obvious necessity for reducing the core loss in transformers to the lowest attainable limit, it is now the custom to carefully test the iron to be used and to express the determined hysteresis loss in the iron sheet, in watts per pound per 100 frequency, at a maximum induction during the cycle of 2500 C.G.S. units. Two processes are in ordinary use for effecting this measurement. The most accurate but most tedious is by the employment of a ballistic galvanometer and the graphical delineation of a number of (B, H,) cycles. This method has the advantage that it can be conducted with either small or large samples of iron, but the reduction of the observations to give the required hysteresis loss is very tedious ; and including the construction of the necessary ring-coil, the actual observations, and the reductions, a single determination of the hysteresis value of an iron can hardly be carried out, even by a couple of industrious workers, in less than a day.

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[*] Read June 11, 1897.

The method employed by Professor Ewing and Mr. F. Holden of rotating a laminated specimen of the iron in a magnetic field, or rotating a field round the specimen and comparing the torque produced on the field-magnet or on the specimen with that produced by a similar sample of iron of known hysteretic value or else measuring it by a calibrated torsion spring, is very much more rapid. This last method is, however, open to the great objection that if used merely to compare a sample of iron with a standard sample, the user of the instrument is at the mercy of his instrument-maker. He receives with it one or more samples of iron which he is told have certain hysteretic values, but he cannot check the truth of the statement. Moreover, if he happens mechanically to injure the standard samples he may seriously change their hysteresis constant. The writer therefore sought for a method which should combine the advantages of both the above processes whilst eliminating their disadvantages, and which should be an absolute method, like the ballistic method, but at the same time a rapid method like the rotating-field method.

The following paper contains a description of a process which is based upon the use of the bifilar reflecting wattmeter and operates upon samples of iron, large or small, in the form of straight strips which are simply slipped into a long magnetizing coil. All the actual practical work and the numerous calculations involved in putting this process to test have been conducted under the author's direction by Mr. A. R. Peart and Mr. W. M. Park working in the Electrical Laboratory of University College, London ; and to these gentlemen belongs the full credit for all this portion of the research as well as for the labour required in much tedious numerical and graphical work.

The method is based upon an interesting discovery with regard to the distribution of the induction in a bundle of straight iron strips or iron or steel wires, when placed in a uniform magnetic field.

If a long, straight, magnetizing coil is prepared by winding insulated wire upon a paper or vulcanized fibre tube, and if the length of the solenoid is, say, ten times its mean diameter, then for a distance equal to at least three-quarters of the total length of the coil we have a practically uniform magnetic field in the interior of the bobbin, when a constant current is sent through the wire. If into this uniform magnetic field a bar of iron or straight bundle of iron strips or iron wire is introduced, the length of the iron being not greater than the uniform portion of the interior field of the coil when the iron is not there, this iron has magnetic induction created in it which varies in density from point to point in the bar, but is symmetrical in value about the central point of the bar.

The induction density (B) is greatest in the centre of the bar but falls off towards each end. If the magnetizing coil is traversed by an alternating current, and if the iron is laminated, the maximum value of the induction density in the bar can be easily determined at any point in its length by placing upon the bar a flat bobbin of fine insulated wire which is connected with an electrostatic voltmeter capable of reading from 1 to 20 volts (see diagram). The first portion of this investigation was concerned with the study of the mode in which this induction density varies from point to point in the length of a bundle of iron strips or iron wire or steel rods of various lengths and diameters, thus placed in the centre of a long magnetizing solenoid.

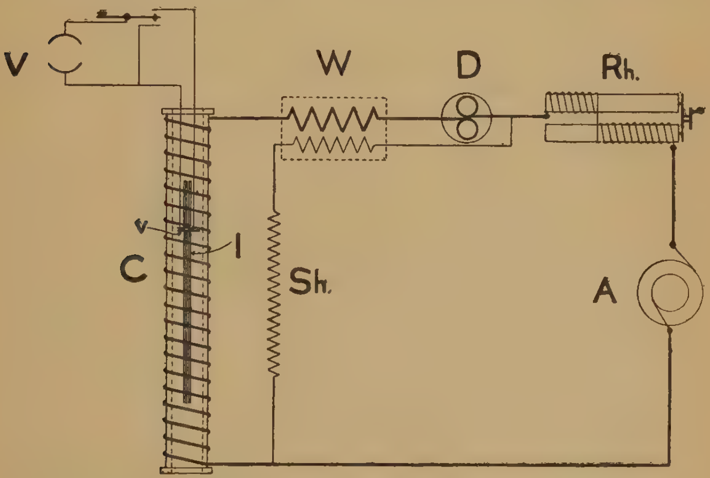
A magnetizing coil was prepared, wound on a paper tube, 5·7 centimetres in internal diameter and 99 centimetres long. On this tube was wound, in three layers, a coil of 1235 turns of cotton-covered copper wire, ·064 inch diameter or of No. 16 size S.W.G. The resistance of this coil was 1·069 ohms at 15° C.

The number of turns per centimetre length of this coil was 12·47. Hence since the interior central field is $4\pi/10$ times the ampere turns per unit of length of the coil, we have thus the value of the field H in the central regions of the interior of this bobbin given by the equation

$$H=15\cdot66\,A,$$

where A is the ampere current through the coil.

The secondary or exploring coil consisted of a flat bobbin wound up of silk-covered copper wire ·0024 inch in diameter



C. *Magnetizing coil.* I. *Iron strips.* v. *Exploring coil.*
V. *Electrostatic voltmeter.* W. *Wattmeter.* D. *Dynamometer.*
Rh. *Rheostat.* A. *Alternator.*

and having 2000 turns; the thickness or length of the bobbin being 0·7 centimetre and the external diameter about 4·5 centimetres.

The exploring coil was wound on a circular ebonite former having a rectangular hole in it so as to closely fit the bundle of iron strips to be used, but at the same time it could be slid along the bundle.

The voltmeter used with it was one of Professor Ayrton's low-reading electrostatic instruments of the idiostatic type and read from 1 to 25 volts, the reading being taken by a mirror and scale. The voltmeter was calibrated when required by means of standard cells or a potentiometer.

The first experiments were conducted with strips of transformer iron, kindly furnished to us by Messrs. Sankey. This sheet iron was ·0121 of an inch or ·0307 of a centimetre in thickness and the strips were each one inch or 2·54 centimetres wide. The strips used were of various lengths: 60 centimetres, 30 centimetres, 15 centimetres, down to 5 centimetres. A bundle I (see diagram) of a certain number of strips was made and the secondary coil v slipped on to it. The magnetizing coil C was then traversed by an alternating current having a frequency of 80 ∼, and the R.M.S. (root mean square) value of the current was read by a dynamometer D or Kelvin balance.

The electrostatic voltmeter V being connected to the ends of the secondary coil gave the R.M.S. value of the induced electromotive force set up in the exploring coil. The form of the curve of the primary current was very nearly a true sine curve, and the variation of induction density in the iron at any point closely followed the same law.

In any case, if f is the *form factor* * of the secondary voltage curve, which in these experiments was 1·13, and if

N = the number of windings in the secondary coil,
S = the total cross-section of the iron,
B = the maximum value of the induction density in the iron in C.G.S. units at that point in its length at which the secondary coil is placed, and
n = the frequency of the induction cycle; then if e_2 is the R.M.S. value of the secondary electromotive force in volts as measured by the voltmeter, we have always

$$10^8 e_2 = 4 f N S B n.$$

Hence the maximum value of the induction density B can be calculated from the known value of all the other quantities. In this manner a large number of measurements of the value of the induction density B were made at different points in the lengths of various bundles of iron strips of different lengths and total cross-sections.

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* See 'The Alternate Current Transformer,' Fleming, vol. i., new. ed., p. 583, for a reference to this term.

These values were then laid down in a series of induction density curves so drawn that the ordinates represented the maximum value of the induction density in the bar at different transverse sections along the half-length of the bar, taken as corresponding abscissæ.

It has been shown by Steinmetz and fully confirmed by others that for most varieties of iron used for electrotechnical purposes the hysteresis loss in the iron, per cycle of induction, per cubic centimetre varies as a power of the maximum value of the induction density, which is very nearly 1·6, when that maximum value of the induction density has any value below about 9000 C.G.S. units. Hence if the ordinates of the above-mentioned induction curves are all raised to the 1·6th power and a new curve plotted showing the variation of $B^{1\cdot6}$ along the half-length of the bar, we have another curve which represents the variation of the hysteresis loss per unit of volume along the half-length of the bar from point to point. A large series of such curves showing the variation of B and $B^{1\cdot6}$ were drawn for bundles of iron strips and steel of different lengths and sections, and the results are collected and represented in the form of curves in the diagrams in Plate VII.

The upper curves in each case in the figs. 1, 2, and 3 in Plate VII. represent the variation of B along the half-length of the bar, and the lower curves the corresponding curves for the variation of $B^{1\cdot6}$. The first pair of diagrams represent the result of varying the maximum induction density in the centre of a bundle of strips of iron of the same length and section.

The second pair of diagrams give the curves showing the same results for bundles of iron strips of varying lengths but the same section.

The third pair for various samples of iron strips of the same length but different section.

It is clear from these diagrams therefore that such a bar or bundle of iron strips placed in an originally uniform magnetic field has a non-uniform induction density produced in it at its various points, which is symmetrical about the centre ; and a non-uniform hysteresis loss per unit of volume also symmetrical about the centre. The maximum values of each being at the centre of the bar.

If we take the true mean value of the varying distributed hysteresis loss per unit of volume, it is obvious that at some point in the half-length of the bar there must exist an actual induction density B_1 such that the true mean hysteresis loss in the whole bar is proportional to $B_1^{1 \cdot 6}$. Let this value of the induction density be called the *effective value*, and the point in the bar at which it occurs the *effective point*. Then the effective value B_1 must be equal to the 1·6th root of the mean of the 1·6th powers of the actual induction densities taken at equidistant points all along the bar, or

$$B_1 = \sqrt[1 \cdot 6]{m \, . \, B^{1 \cdot 6}}$$

where $m \, . \, B^{1 \cdot 6}$ stands for the mean ordinate of the curve representing the varying values of $B^{1 \cdot 6}$ all along the half-length of the bar.

If then this effective value B_1 of the induction density is calculated for each of the samples of iron strip used, and the position at which the actual induction has this effective value is marked off on the half-length of the bar (this position is represented by the *cross* on the curves given in Plate VII.), we have noticed the remarkable fact that *whatever the length or section of the iron strip may be, the point in the bar at which the actual induction density has a value equal to the effective value, always comes at the same proportional distance from the centre of the bar ; and this distance is very nearly equal to ·56 of the half-length of the bar from the centre or at ·22 of the whole length of the bar from one end.*

The following Table I. embodies all the results taken with bundles of annealed transformer iron strip (Sankey's) of very different lengths and sections, and Table II. similar observations on rods of steel and hard iron ; and it will be seen that the effective induction in the case of the iron strip is always found at a point in the bar or bundle which is very nearly 0·56 of the half-length of the bar from the centre. If therefore the secondary coil is placed at that spot and the secondary voltage then observed used to calculate the induction density, the value so obtained is that which corresponds to the true mean value of the varying hysteresis loss per unit of volume all along the bar. In other words, if we suppose the bar magnetized by an alternating current uniformly and with a maximum induction density everywhere equal to that which it actually has under normal conditions at 0·56 of the half-length of the bar from the centre; then the hysteresis loss in such a bar would be equal to that found in the bar as it is. In all cases where induction density values are given, these are to be understood as being the maximum values during the cycle unless otherwise stated. The frequency of alternations used was 80.

We have therefore in this useful fact regarding the distribution of induction in iron strips in a uniform magnetic field, the foundation of a quick method of determining the hysteretic constant of a sample of iron.

If the iron sample, taken in strips, is placed as above in the interior of the long magnetizing coil, and if a sensitive wattmeter, suitable for use with alternating currents, is connected therewith, we can measure the total power taken up in the iron and in the copper of the coils when any known alternating current is passing through the magnetizing coil. If then the iron strips are withdrawn and another measurement of the same kind made with the current passing through the magnetizing coil adjusted to the same value as in the first case, this last reading gives us the wattmeter-reading corresponding to a known power taken up in the copper of the magnetizing coils. Hence the first wattmeter-reading gives us the iron loss, together with the copper loss, and the second reading gives us the copper loss alone. The iron loss gives us the true mean value of the hysteresis loss in the iron, assuming the iron is so well laminated that eddy-current loss is practically absent.

The above-mentioned law as to the distribution of the induction enables us, then, to assign the corresponding value of the maximum induction density.

Unless this can be done, the mere mean hysteresis loss measurement is no practical use *.

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* Methods of measuring hysteresis loss by one observation of an instrument have been suggested by other observers; but the practical value of these methods has been very small, because the investigators have not shown how the corresponding maximum value of the effective induction is to be obtained. The rule above given supplies this deficiency.

TABLE 1.

Summary of experiments made to ascertain the "Effective point" in bundles of annealed Transformer Iron Strips (Sankey's) when placed in a uniform alternating magnetic field.

Reference letter to diagram.	Length of specimen in centims.	Sectional area of specimen in sq. cms.	Induction at centre of bar, B. C.G.S. units.	Effective induction, B_1. C.G.S. units.	Ratio of $\frac{B_1}{B}$.	Distance of effective point from centre of bar.	Fraction of half-length of bar from centre at which effective induction exists.
A_1	61	·936	564	420	·744	17·01	·558
A_2	61	·936	1648	1240	·752	16·91	·554
A_3	61	·936	2084	1563	·750	16·92	·555
A_4	61	·936	2536	1900	·749	17·00	·557
A_5	61	·936	3112	2335	·712	17·00	·557
A_6	61	·936	3816	2870	·752	17·10	·561
B_1	30	·936	1597	1192	·746	8·36	·557
B_2	30	·936	4052	3012	·743	8·35	·557
B_3	15	·936	1622	1265	·780	4·11	·548
B_4	15	·936	3430	2660	·775	4·18	·557
B_5	10	·936	1907	1554	·815	2·73	·546
B_6	10	·936	1025	833	·813	2·76	·552
B_7	5	·936	1622	1420	·875	1·39	·556
B_8	5	·936	778	678	·871	1·49	·596
C_1	61	·312	3485	2565	·736	16·84	·552
C_2	61	·312	9673	7250	·749	16·81	·551
C_3	61	·624	2125	1570	·739	17·07	·560
C_4	61	·624	5834	4420	·758	17·60	·577

Mean of last column = ·56.

TABLE II.

Summary of tests made as in Table I. upon Steel and Hard Iron Rods.

Reference letter to diagram.	Length of specimen in centims.	Sectional area of specimen in sq. cms.	Induction in centre of bar, B. C.G.S. units.	Effective induction, B_1. C.G.S. units.	Ratio of $\frac{B_1}{B}$.	Distance of effective point from centre of bar in cms.	Fraction of half-length of bar from centre at which effective induction exists.	Nature of bar.
C_6	61	·970	4499	3525	·783	6·90	·58	Round bar of silver steel $\frac{7}{16}$ inch in diameter.
D_2	32	·178	5997	4784	·798	9·80	·61	Round rod of silver steel $\frac{3}{16}$ inch in diameter.
D_3	32	·534	5555	4244	·764	9·31	·58	Three similar pieces of steel.
D_4	20·45	·645	4746	3587	·756	6·67	·65	Bundle of 60 iron wires of hard iron of rectangular section.
D_5	20·45	·645	2545	1912	·751	6·76	·66	

The operation of making the mean hysteresis measurement is effected by a sensitive mirror bifilar electrodynamometer.

This was constructed in the following way:—

The fixed or series-coil consists of 90 turns of No. 16 S.W.G. cotton-covered copper wire wound in five layers on a rectangular block of wood. The movable coil, or shunt-coil, consists of a rectangular coil of 20 turns of No. 24 S.W.G. wire, the turns being fastened together with silk and shellac. This movable coil embraces, without touching, the fixed coil, as in a Siemens dynamometer. The movable coil is suspended by a bifilar suspension consisting of two silver wires ·002 of an inch in diameter and 37 centim. long, fixed about 3 millim. apart. These wires are attached to the ends of the suspended coil, and serve to convey the shunt-current in and out of the coil. The movable coil carries a concave silver mirror by means of which a sharp image of an incandescent lamp-filament is formed upon a scale at a fixed distance of 100 centim. from the mirror. A torsion-head carries the upper ends of the bifilar suspension, and serves to adjust the zero.

In series with the shunt-coil is a high non-inductive resistance of 1000 ohms.

The resistance of the thick-wire fixed coil of the wattmeter is ·133 of an ohm, and the resistance of the shunt-coil and suspension, without the associated inductionless resistance, is 11·7 ohms.

When such a bifilar wattmeter is employed to measure electrical power, it can be shown that the deflexions of the movable coil, when not large, are strictly proportional to the power passing through the wattmeter, so that each scale-division has an assignable power value, which in our case was almost exactly ·01 of a watt per millimetre of scale. The arrangement of circuits was then as shown in the diagram on p. 205. The long magnetizing coil above described was connected in series with the fixed coil of the wattmeter, and also with a sensitive electrodynamometer or ammeter suitable for measuring small alternating currents of about an ampere or less in value.

The shunt-coil of the wattmeter was joined across the ends of the long coil. Suitable resistances beyond controlled the current.

The process of measurement was then as follows:—A bundle of straight iron strips was fitted with the secondary coil at the effective point, viz. ·56 of the half-length of the bar from the centre. The bundle was slipped into the magnetizing coil, and an alternating current sent through the coil of such a magnitude (generally about ·2 of an ampere) as to give an induction density of not more than about 6000 C. G. S. units at the effective point. The wattmeter deflexion was then read. To interpret this reading the iron strips were removed from the coil and another wattmeter measurement made on the copper coil alone. As the removal of the iron sends up the current strength by decreasing the inductance of the circuit, it is necessary to readjust the current to the same value as before, and to know that value in amperes. The resistance of the coil being known, and also the current passing through it, we can calculate the power being taken up in the copper coil alone for this standard current; and hence know the power in watts producing the observed wattmeter deflexion.

The wattmeter is therefore calibrated on the copper circuit of the magnetizing coil, and from the known copper loss in this coil and the proportionality of its deflexions we are enabled to assign at once the value in watts of its deflexion when used to measure both the iron loss and copper loss together when the iron strip is inserted in the coil.

If the induction is carried up as far as 4000 or 5000 C.G.S. units at the effective point, and if the strips have a thickness over ·01 inch, then it becomes necessary to apply a small correction for the eddy-current loss set up in the iron. This can be at once done by using the following simple practical formula given by the author * for this loss in rectangular sectioned strips.

If t is the thickness of the strip in *mils* (1 mil = ·001 inch), n is the frequency of the current, and B is the maximum value of the induction density during the cycle at any point on the bar, then it can be shown that the eddy-current loss in watts per cubic centimetre at that point in the rectangular or sectional strip is closely given by the expression :—

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* See 'The Alternate Current Transformer,' Fleming, vol. ii. p. 490, for proof of this formula.

$$\left(\frac{B}{10^8}nt\right)^2,$$

and hence for a given frequency and thickness the eddy-current loss per pound of iron is very quickly found from a simple parabolic curve representing B^2 in terms of B.

In all the above observations with the iron strip used, which was 12 mils thick, this correction has been applied, and it is sufficient for this purpose to take as B in the above formula the value B_1 of the effective induction.

The eddy-current loss varies from point to point in the bar, being always proportional to the square of the maximum value of the induction density. Hence, properly speaking, the mean value of the eddy-current loss should be obtained by taking the mean square value of the induction all along the half-length of the bar, and using the mean-square value instead of the B^2 in the formula above.

We have, however, found that the point on the half-length of the bar at which the actual induction has a value equal to the root-mean-square value, or to the square root of the mean of the squares of the distributed induction, is always very close to the point at which the 1·6th root of the mean of the 1·6th powers of the distributed induction is found. In other words, the induction at the effective point is the induction corresponding to the mean eddy-current loss as well as to the mean hysteresis loss. This is shown by the curves given in figs. 4 and 5, Plate VII., in which the curve marked B is the curve of variation of the maximum value of the induction-density along the half-length of a bundle of iron strips, and the curves marked $B^{1\cdot6}$ and B^2 represent (to different scales) the variation of these powers of B. In a bar 5 centim. long (see fig. 5, Plate VII.) the points where the actual induction had a value equal to $\sqrt{m\,.\,B^2}$ and to $\sqrt[1\cdot6]{m\,.\,B^{1\cdot6}}$ were only separated by a fraction of a millimetre. Hence by another curious coincidence we are enabled to apply the proper correction for the eddy-current loss by taking the square of the effective induction B_1 and putting its value into the expression

$$\frac{B_1{}^2 n^2 t^2}{10^{16}},$$

where n is the frequency and t the thickness of the strip in mils.

In order to put the method to a test, a very careful determination was made by the ballistic method of the hysteresis loss of a sample of the transformer iron strip used. A ring-coil was prepared, and a series of (B-H) cycles of magnetization taken, and a hysteresis loop drawn, and from this loop it was found that for this iron the hysteresis loss was ·25 watts per pound per 100 cycles per second, for a maximum value of the induction density equal to 2500 C.G.S. units.

Straight strips of exactly the same iron were then tested by the method here described, which for shortness will be called the "straight coil" method, and the results of these tests with the iron in different conditions of hardness are shown in the Tables III., IV., and V.

It will be seen that the results of the "straight coil" test with the wattmeter, as given in Table III., gave exactly the same mean value for the hysteresis loss of the strip when not bent and unbent, as did the calculations made from the ballistic test, viz.: ·25 watts per pound per 100∼ per maximum induction density of 2500. The difference in the time of taking the tests is, however, immense. A very few minutes suffices to take the wattmeter tests, and the necessary calculations can be made in a comparatively short time.

Many dozens of tests of irons can be made by the straight coil method in the time occupied by one ballistic test.

There is another method by which the bifilar wattmeter can be calibrated, but which is not so generally convenient as that above described. If the shunt and series coils of the wattmeter are joined up in series with each other, so as to form a dynamometer, and if a known current I is passed through these coils, causing a deflexion θ in the movable coil; then when the instrument is used as a wattmeter with a total shunt-resistance R (including the added non-inductive resistance) the true power corresponding to a deflexion θ is I^2R watts.

Hence, from a curve showing the deflexion given by various currents, we can at once determine the constant of the instrument, when used as a wattmeter, with given shunt resistance.

If this standardization is taken with continuous currents, then it is necessary to eliminate the effect of the Earth's field by taking right and left deflexions. The curve connecting square of current strength and the deflexions of the movable coil is nearly a straight line, in the case of the bifilar wattmeter.

Standardizing our wattmeter in this way with a continuous current, we then made another set of hysteresis measurements, with 12 strips of perfectly new annealed transformer strips (Sankey's), with the results shown in Table VI. Two other sets of observations for hammered and hardened iron are given in Tables VII. and VIII.

As a matter of fact, in the above experiments in the observations given in Tables III., IV., V., VII., and VIII., the wattmeter was standardized by sending known and measured currents through the copper coil and obtaining a wattmeter constant by finding the deflexions corresponding to known copper losses in the copper coil. Thus are found the following figures, when using the wattmeter to measure power taken up in the magnetizing coil alone :—

Known copper loss (C^2R) calculated from observed current and coil resistance in watts.	Scale-deflexions of wattmeter in millimetres.	Watts per millimetre of scale-deflexion of wattmeter.
·398	39·1	·0102
·292	28·9	·0101
·197	19·0	·0103

TABLE III.

Hysteresis Test by "Straight Coil" method of annealed Transformer Iron Strip. Six strips used (new) 61 cm. long, 2·54 cm. wide, and 12 mils thick.

Voltmeter scale-reading in mm.	Effective Induction density, B_I.	Wattmeter scale-reading.		Total iron loss in watts per lb.	Eddy-current loss in watts per lb.	Hysteresis loss in watts per lb. at 80∼.	Hysteresis loss in watts per lb. at 100∼ and 2500 B.	Mean hysteresis value.
		Iron in.	Iron out.					
303·0	6719	87·6	17·5	1·247	·245	1·002	·257	·25 watts per pound per 100 ∼per 2500 B.
225·3	5702	66·0	15·3	·9027	·176	·727	·243	
131·8	4284	40·0	7·8	·5747	·099	·476	·251	
95·1	3623	43·0	8·1	·4377	·071	·367	·251	

TABLE IV.

Hysteresis Test by "Straight Coil" method of Transformer Iron Strip. Same strips used as in Table III., but once bent and unbent.

Voltmeter scale-reading in mm.	Effective Induction density, B_1.	Wattmeter scale-reading.		Total iron loss in watts per lb.	Eddy-current loss in watts per lb.	Hysteresis loss in watts per lb. at 80∼.	Hysteresis loss in watts per lb. at 100∼ and 2500 B.	Mean hysteresis value.
		Iron in Coil	Iron out of Coil.					
56·9	2786	19·0	4·2	·303	·042	·261	·273	·28 watts per pound per 100 ∼ per 2500 B.
128·8	4244	39·0	8·4	·635	·098	·537	·288	
204·0	5424	58·1	12·4	·948	·160	·788	·285	
273·0	6346	76·0	15·9	1·247	·219	1·028	·289	

TABLE V.

Hysteresis Test by "Straight Coil" method of Transformer Iron Strip, the same quality as in Tables III. and IV., but bent and unbent several times. Twelve strips used 30 cm. long, 12 mils thick, cut from the same pieces used for the ballistic galvanometer test.

Voltmeter scale-reading in mm.	Effective Induction density, B_1.	Wattmeter scale-reading.		Total iron loss in watts per lb.	Eddy-current loss in watts per lb.	Hysteresis loss in watts per lb. at 80∼.	Hysteresis loss in watts per lb. at 800∼ and 2500 B.	Mean hysteresis value.
		Iron in coil.	Iron out of coil.					
374·5	3778	135·7	105·1	·539	·078	·461	·296	·28 watts per lb. per 100∼ per 2500 B.
281·4	3219	101·4	78·1	·415	·057	·358	·297	
199·5	2672	71·2	54·8	·277	·038	·239	·268	
111·6	1964	42·1	32·9	·160	·0215	·139	·252	
71·1	1625	57·0	43·1	·125	·017	·107	·282	
50·4	1314	39·2	29·9	·083	·011	·072	·261	

TABLE VI.

Hysteresis Test by "Straight Coil" method of Transformer Iron Strips. Twelve new annealed strips used, never bent. Wattmeter standardized by Potentiometer and Continuous Currents.

Voltmeter scale-reading in mm.	Effective Induction-density, B_1.	Wattmeter scale-reading.		Total iron loss in watts per lb.	Eddy-current loss in watts per lb.	Hysteresis loss in watts per lb. at 80∼.	Hysteresis loss in watts per lb. at 100∼ and 2500 B.	Mean hysteresis value.
		Iron in.	Iron out.					
497·0	4742	90·6	26·0	·517	·118	·399	·178	·21 watts per pound per 100 ∼ per 2500 B.
338·8	3576	63·6	17 1	·340	·069	·271	·190	
267·0	3136	50·7	14·7	·295	·053	·242	·210	
204·9	2715	39·8	11·7	·232	·038	·193	·210	
130·8	2138	51·6	14·6	·154	·025	·129	·208	
110·3	1957	44·0	11·9	·133	·022	·111	·205	
77·1	1628	31·0	8·7	·093	·016	·070	·191	
49·5	1295	21·1	6·1	·062	·010	·052	·187	
398·5	3913	71·0	17·5	·466	·083	·383	·234	
282·0	3223	100·9	25·1	·334	·056	·278	·230	
200·5	2679	73·2	18·1	·242	·039	·203	·226	
100·0	1950	42·9	11·0	·118	·021	·097	·181	
77·2	1627	61·4	16·1	·103	·016	·087	·217	
49·2	1302	41·2	11·7	·067	·010	·057	·203	

N.B. The method of standardizing the wattmeter by use as a dynamometer employing a continuous current as above does not give such consistent or accurate hysteresis determinations as when it is standardized on a known copper resistance with known alternating currents.

TABLE VII.

Hysteresis Test by “ Straight Coil ” method on Hammered Transformer Strip (unannealed). Six strips used 61 centim. long, 2·54 centim. wide, and 12 mils thick.

Voltmeter scale-reading in mm.	Effective induction-density, B_1.	Wattmeter scale-readings.		Total iron loss in watts per lb.	Eddy-current loss in watts per lb.	Hysteresis loss in watts per lb. at 80∼.	Hysteresis loss in watts per lb. at 100∼ and B 2500.	Mean hysteresis value.
		Iron in.	Iron out.					
114·0	3975	54·5	23·5	·562	·086	·476	·284	·290 watts per pound per 100 cycles at B 2500.
160·0	4737	72·9	30·2	·778	·123	·655	·297	
206·0	5438	90·5	37·5	·969	·161	·802	·290	

Calibration Table of Wattmeter.

C^2R.	Wattmeter deflexion.	Watts per millim.
·1875	20·9	·00897
·3263	36·9	·00884
·1195	13·3	·00898

TABLE VIII.

Hysteresis Test by “Straight Coil” method of 18 strips of Iron of inferior quality. Length 61 cms., width ·63 centim., thickness 26·5 mils.

Voltmeter scale-reading in mm.	Effective induction-density, B_1.	Wattmeter scale-reading in mm.		Total iron loss in watts per lb.	Eddy loss in watts per lb.	Hysteresis loss in watts per lb. at 80∼.	Hysteresis loss in watts per lb. at 100∼ and B 2500.
		Iron in.	Iron out.				
83·4	2054	51·9	23·7	·319	·110	·209	·360
136·0	2644	79·7	34·1	·517	·183	·334	·360
176·0	3028	101·0	42·8	·659	·239	·420	·386

Calibration Table of Wattmeter.

C^2R.	Wattmeter deflexion in mm.	Watts per mm.
·272	30·0	·00908
·381	41·4	·00920

In the case of the observations given in Table VI., we employed the method of determining the wattmeter constant by passing known currents through its coils joined up in series and used as a dynamometer, and from the curve showing relation of (current)^2 to deflexions we could at any moment find the wattmeter constant.

The method described above has this great advantage that the test can be made on fairly large samples of iron, and it can be made with an alternating current having any desired and practical form factor. The iron is tested under conditions which are exactly similar to those under which it will be used in transformer manufacture, and, in fact, the straight magnetizing coil and iron sample constitute an open circuit transformer of very small power factor.

The above described method effects a great saving of time in comparison with the ballistic method, whilst at the same time it is an absolute method and does not depend upon comparisons with other iron samples of supposed known hysteretic value.

In converting hysteresis losses from one maximum induction-density B to another by the aid of Steinmetz's law, it is useful to have the arithmetic values, for different values of B, of the ratios of $B^{1\cdot6}$ to $2500^{1\cdot6}$; since B = 2500 C.G.S. units is generally taken as the standard induction-density in reference to hysteresis determinations. The following table (Table IX.)

Table IX.

Showing the Values of the Ratio of $B^{1\cdot6}$ to $2500^{1\cdot6}$.

B.	$\frac{B^{1\cdot6}}{2500^{1\cdot6}}$.	B.	$\frac{B^{1\cdot6}}{2500^{1\cdot6}}$.
1000	·231	5500	3·531
1500	·442	6000	4·058
2000	·700	6500	4·614
2500	1·000	7000	5·193
3000	1·338	7500	5·795
3500	1·713	8000	6·430
4000	2·118	8500	7·085
4500	2·563	9000	7·764
5000	3·031	9500	8·466
		10000	9·190

gives these ratios with sufficient accuracy for all practical work. Thus, for instance, if the value of the hysteresis loss in an iron has been determined at a maximum induction-density during the cycle of 5000 units, and we wish to know the value at 2500 units, the observed value has to be divided by 3·031 to obtain the required result.

Discussion.

Mr. Carter asked whether roots other than the 1·6th gave a similar constant value of the induction density.

Dr. Fleming said it seemed to be the result of accident that the 1·6th root gave a constant value for iron.

XX. *Attenuation of Electric Waves along Wires and their Reflexion at the Oscillator.* *By* Edwin H. Barton, *D.Sc., F.R.S.E., Senior Lecturer in Physics at University College, Nottingham*[*].

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Last year Mr. Bryan and the present author realized experimentally the absorption by a terminal bridge of the electric waves incident upon it[†]. The first application of this result which suggested itself was the use of such a bridge in determining the attenuation (or rate of decay) of the waves along their parallel leads. For, if one can at pleasure either (1) practically absorb the waves, or (2) reflect them practically undiminished, a suitable electrometer being inserted in the line, we have in the first case a single passage of the wave-train past the electrometer, and in the second a number of passages of the wave-train until it is practically quenched by its repeated losses. Thus, the electrometer in the second case is affected by the sum to infinity of two geometrical progressions, viz., those due to the forward and return waves respectively. Whereas in the first case, when the

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[*] Read June 25, 1897.

[†] Proc. Phys. Soc. vol. xv. pp. 23–30, 1897; Phil. Mag. [5] xliii. pp. 39–45, 1897.

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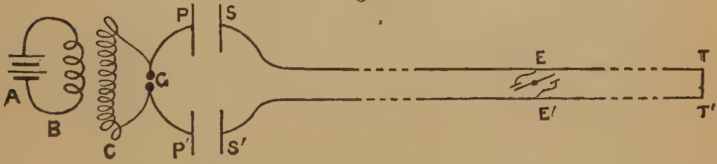
waves are absorbed at the end of the line, the electrometer is affected by the first term only of one of the above series. Hence the ratio of the electrometer-throws in the two cases is a function of the attenuation constant and the dimensions of the line, and would appear to be immediately calculable. But this is not all. For the above is on the assumption that the losses occur only along the wires themselves, whereas something is lost also on reflexion at the primary oscillator where the waves are generated. This fact soon became apparent on the repetition of the experiment with different lengths of line, and the value of this reflexion coefficient was substantially confirmed by further experiments and by the application of this new view of the case to experiments previously carried out in Bonn.

Experimental Arrangement.—The arrangement of the apparatus adopted in the determination is diagrammatically represented in fig. 1. In this figure, A denotes the battery

Fig. 1.



of two storage-cells, B and C are the primary and secondary coils respectively of the induction-coil, of which C has a resistance of 3000 ohms and an inductance of about 20 henries. G is the spark-gap, which was adjusted to 2 mm. The wire PGP′, measured along the semicircle, was about 2 m. PP′ are condenser-plates of zinc 40 cm. diam. placed opposite to and 30 cm. distant from the precisely similar plates S S′. The line is represented by S E T T′ E′ S′, and consists of two parallel copper wires 1·5 mm. diam., and kept 8 cm. apart by wood separators at intervals of about 2 m. E E′ denotes the electrometer, which has a single plane needle, initially uncharged, and suspended by a fine quartz fibre between two disks attached to the line at E and E′. The needle is therefore electrified by induction whenever a wave passes E E′, and its ends are consequently attracted to the disks whatever the sign of their potential-difference. T T′ signifies the several pieces of apparatus successively used at or near the end of the line, and will be described in detail for each case when dealt with. The lengths of the line before and after the electrometer will also be stated in connexion with each experiment. The waves generated by the primary oscillator were about 8·5 m. long.

Theory.—Let δ_1 be the electrometer-throw obtained when a completely absorbing bridge is used at T T′ (fig. 1), and let δ_2 be the throw when a "no-resistance" bridge is used at T T′, *i. e.*, a piece of short thick copper wire, which therefore reflects the waves without appreciable loss. Then it is required to express the ratio δ_2/δ_1 as a function of the attenuation and reflexion coefficients and then, from the experimental values of the above ratio, to solve the resulting equations for the coefficients sought.

Let the maximum potential-difference of the two wires of the line due to any wave be taken as the amplitude of that wave, let ρ be the factor by which the amplitude of a wave is affected on reflexion from S S′ (fig. 1), and let $e^{-\sigma x}$ be the attenuation factor by which the amplitude is affected in traversing x cms. of the line. For convenience of actual working $10^{-sx'}$ was substituted for the above, x' denoting length traversed in metres, σ was then deduced from s.

Now the throws of the electrometer are proportional to the time-integral of the square of the amplitude of the passing wave-train, *i. e.*, proportional to its energy: hence, if the lengths of the line before and after the electrometer are l_1 and l_2 respectively, and a wave-train of initial energy i leaves the oscillator, we have with the "absorber" at the end of the line

$$ie^{-2l_1\sigma}=k\delta_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where k is the electrometer constant.

Again, with the no-resistance bridge or short circuit at the end, we have two series of impulses at the electrometer, the energy of the forward waves being

$$ie^{-2l_1\sigma}[1+\rho^2e^{-4(l_1+l_2)\sigma}+\ldots],$$

and that of the backward ones $[e^{-4l_2\sigma}]$ times the above. Hence we obtain

$$\frac{ie^{-2l_1\sigma}(1+e^{-4l_2\sigma})}{1-\rho^2e^{-4(l_1+l_2)\sigma}}=k\delta_2, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

whence

$$\delta_2/\delta_1=\frac{1+e^{-4l_2\sigma}}{1-\rho^2e^{-4l\sigma}}=r \text{ say}, \quad . \quad . \quad . \quad . \quad (3)$$

l being written for l_1+l_2.

Two experiments with different values of l_1 and l_2 furnish two equations like (3); these are then cast in the form

$$\rho^2re^{-4l\sigma}=r-1-e^{-4l_2\sigma}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and ρ^2 eliminated between them. The result was an equation in σ of the form

$$A_1e^{-a_1\sigma}+A_2e^{-a_2\sigma}+\ldots=0,$$

or say

$$u=0.$$

Various trial values were then given to σ and plotted as abscissæ of a curve, the corresponding values of u being its ordinates. The value of σ satisfying the equation was thus found graphically as that at which the curve cut the axis of abscissæ.

Experiments.—In the experiments the bridge used as an absorber was placed absolutely at the end of the line, as it is only when in that position that it can absorb all. It consisted of pencil marks on a ground glass disk as previously described, and before use each time was carefully adjusted to the requisite resistance for absorption as mentioned in the paper already referred to. A doubt has been expressed as to the constancy of the value obtained for a resistance of this kind by the Post-Office box, some similar resistances having been said to measure differently according to the voltage used in the test. Probably this might be the case to a slight extent if extreme variations were used, but with the pencil-mark resistances employed in this work no change of value was observable, whether a single Leclanché cell was used or a battery of several storage-cells, so that the resistance was determined with all the accuracy needed for the present purpose.

The other bridge, intended to reflect undiminished the waves incident upon it, was placed a little short of the end of the line, and occupied alternately two positions a quarter of a wave-length apart. This precaution was adopted lest, far as the bridge was beyond the electrometer, the head of the wave-train reflected from the bridge might reach the electrometer before the tail had passed it. In this case a slight interference would arise at the electrometer, and its throws would not be of the values due to simple passages and re-passages of the wave-train. By using the above two positions for the bridge this source of error is obviated, for, if one gave additive interference at the electrometer, the other would give about equal subtractive interference, and their mean would be the result desired as though no interference had occurred.

The length l_2 is of course measured from the electrometer to the point midway between these two positions of the bridge.

Further, to eliminate errors due to the unavoidable irregularity of the sparking in the primary oscillator, a number of readings, about 20, with the absorber only at the end of the line were alternated with a like number of readings with the reflecting bridge interposed. The separate ratios of the electrometer-throws obtained under the two conditions were then taken, and the probable error of their arithmetic mean calculated by the theory of probabilities.

Observations and Results.—Four such sets of readings were taken with different values of l_1 and l_2. One set is shown in Table I., and the data and results of the whole four summarized in Table II.

It may be interesting to regard, in the light of the results in the second table, two sets of observations taken by the writer in Bonn during the summer of 1893 while working under the late Professor Hertz[*]. In the course of those experiments (directed primarily to an object quite different from

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[*] Proc. Roy. Soc. vol. lvii. pp. 72 & 75 (1895); D.Sc. Thesis for the University of London, 1894, pp. 20–22 and 28–29; Wied. *Ann.* Bd. liii. pp. 525–527 and 528–529 (1894).

Table I., showing readings for $l_1=117\cdot5$ m. and $l_2=48$ m.

Electrometer-throws with Absorbing Bridge only at end of line.		Electrometer-throws with Reflecting Bridge		Ratios of throws, viz.:—		Mean Ratios.
Actual observations.	Interpolated Means.	at 48 m.$-\frac{\lambda}{8}$ beyond the Electrometer.	at 48 m.$+\frac{\lambda}{8}$ beyond the Electrometer.	$\frac{\text{col. 3}}{\text{col. 2}}$.	$\frac{\text{col. 4}}{\text{col. 2}}$.	
11						
	13	43		3·307		
15						
	16·5		44		2·666	2·986
18						
	17·5	42		2·400		
17						
	18·5		36		1·946	2·173
20						
	18	41		2·277		
16						
	15		44		2·933	2·605
14						
	16	33		2·062		
18						
	15		33		2·200	2·131
12						
	12·5	32		2·560		
13						
	12		28		2·333	2·446
11						
	13·5	27		2·000		
16						
	18·5		46		2·486	2·243
21						
	22	55		2·500		
23						
	22		48		2·181	2·340
21						
	20·5	56		2·731		
20						
	20		43		2·155	2·443
20						
	20	57		2·850		
20						
	21		52		2·476	2·663
22						
	21·5	50		2·325		
21						
	21·5		49		2·279	2·302
22						

Whence Final Mean Ratio = $2\cdot43\pm0\cdot05$.

TABLE II., summarizing Data and Results of Experiments.

Lengths of line		Ratio of Throws	The unknown quantities for which the equations were actually solved.		Attenuation Constant.	Reflexion Coefficient.	Remarks.
before Electrometer	after Electrometer						
$=l_1$.	$=l_2$.	$=r$.	s.	ρ^2.	σ.	ρ.	
metres. 117·5	metres. 20	2·41 ±0·04	0·000564	0·4776	0·0000130	0·69	As these were the best sets of observations the resulting equations were solved simultaneously.
65·0	20	2·744±0·035					
65·0	48	2·64 ±0·18	0·0005	0·53	0·0000115	0·73	This being the worst set of readings (as seen by the large probable error) was treated alone. Any *number* of pairs of values could, of course, be found that would satisfy the equation.
117·5	48	2·43 ±0·05	0·00043	0·4776	0·000010	0·69	This is the set given in Table I., and is seen to be slightly less trustworthy than those at the head of this table. Two solutions are given of this set, the first seems the more probable.
			0·0005	0·555	0·0000115	0·745	

that now under discussion) the electrometer was placed almost at the end of the line. By an entirely different method from that now adopted, ratios were obtained for the electrometer-throws due (1) to the passage past it of the sum to infinity of the geometrical progressions formed by the outward and return waves, and that due (2) to the single passage of the initial wave-train. The lengths of the line to the electrometer in the two cases were 160 and 91·5 m. respectively, and the corresponding ratios were 1 : 0·34 and 1 : 0·31. Hence for this case equation (4) takes the form

$$\rho^2 r e^{-4l\sigma} = r - 2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

since $l_2 = 0$.

Solving simultaneously the two equations thus derived from the experimental values of r, we obtain

$$\left.\begin{array}{ll} s = 0{\cdot}00027, & \rho^2 = 0{\cdot}476, \\ \sigma = 0{\cdot}0000062 \text{ and } & \rho = 0{\cdot}69 \end{array}\right\} \quad . \quad . \quad . \quad . \quad (6)$$

In these experiments the primary oscillator as regards the parts SPGP′S′ (fig. 1) was precisely the same as in the present paper, and it seems noteworthy that its reflexion coefficient is practically the same. The values of the resistance and inductance of the secondary circuit of the induction-coil used in Bonn cannot be stated, as they were not at the time regarded as of any material consequence and accordingly were not measured.

It will be seen that the attenuation constant in the 1893 case is about half that in the present work. This may possibly be due to the wood separators used along the line being much drier in a German summer than in an English winter and spring.

Comparisons with Heaviside's long-wave theory of reflexion.—After obtaining experimentally the unexpected result that of the wave energy incident upon the primary oscillator only about one-half or less was reflected along the line, it seemed incumbent on one to test this if possible by theory. And in this connexion I have pleasure in acknowledging my indebtedness to Mr. Oliver Heaviside for an explanation of his theory applied to the case of reflexion from a terminal arrangement prior to its publication *.

On Heaviside's theory developed originally for long waves, say of telephonic frequency, we have for the reflexion coefficient

$$\rho=\frac{Z-Lv}{Z+Lv}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where Z is the resistance operator, L is the inductance per unit length of the line, and v is the speed of light.

Now the resistance operator in the case of condensers, secondary and primary coils is given by

$$Z=\frac{1}{Sp}+(R_2+L_2p)-\frac{M^2p^2}{R_1+L_1p}, \quad . \quad . \quad . \quad (8)$$

where S is the capacity of the condenser, the R's are the resistances and the L's the inductances of the coils, M their mutual inductance, and p the time differentiator.

Of the three terms on the right-hand side of (8), the first only is to be used if the arrangement is short-circuited at the spark-gap, G (fig. 1), the first and second only if the primary circuit of the induction-coil is open, but all three if the primary is closed.

Now in the actual experimental case the sparks at the primary oscillator are started by the break of the primary circuit of the induction-coil, the interruptor remaining open during the greater part of its period (say the $\frac{1}{30}$ of a second). But the time required for the wave-train to pass to the end of the longest line used and back again, say a dozen times (which practically extinguishes it), is only of the order 14 millionths of a second. Hence the form of (8) to be adopted as applicable to the present case is

$$Z=\frac{1}{Sp}+R_2+L_2p, \quad . \quad . \quad . \quad . \quad . \quad (9)$$

if indeed the long-wave theory holds at all for waves so short as 8 or 9 m.

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* 'Electromagnetic Theory,' pp. 786–9; 'The Electrician,' April 9, 1897.

But, on working from equation (9) the value of ρ, we obtain a result differing from unity by less than 1 in 10^{10}.

It would appear therefore that the long-wave theory fails to apply to the present case of high-frequency waves and their reflexion from condensers and coils of high inductance, each of which involves of course the time-differentiator.

In the case of a resistance practically devoid of inductance, on the other hand, as was seen in the last paper on the absorbing bridge, the long-wave theory seems to hold within a close approximation.

Reflexion from Condenser only.—However, in order to test the matter experimentally a little further, a second set of plates like PP′ SS′ (fig. 1) was made and put up at the end of the circuit at TT′, but this set had simply one wire in the form of a semicircle to connect the far plates instead of a spark-gap as at G. The induction-coil was also omitted entirely at this end of the line.

Hence the long-wave theory for reflexion for this arrangement gives $Z=1/Sp$ and

$$\rho=\frac{1/Sp-Lv}{1/Sp+Lv}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

And this equation, applied to an incident wave of the type $\phi_1 \sin nt$, gives for the reflected wave $\phi' \sin(nt+\alpha)$, where

$$\left.\begin{aligned} \phi'/\phi=\rho=1, \\ \text{and} \qquad \tan\alpha=\frac{-2LvSn}{1-(LvSn)^2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Here then is an opportunity of comparing simply the behaviour of the short waves with what may rightly be expected of the longer ones. For in this arrangement there is no spark-gap whose heated state in the former case leaves one in some doubt as to how far, when the waves return to it, it acts as a short circuit and so possibly cuts out the induction-coil entirely.

The experiment was now conducted by alternating electrometer readings (1) with the condenser at T T′, fig. 1, and (2) with a no-resistance bridge placed across the wires just before the end of the line.

If then the condenser reflected all, as by equations (10) and (11) must be the case with long waves of the simple harmonic type whatever the capacity of the condenser, we should find the ratio of the electrometer-throws taken under the two conditions to be unity. Whereas the experiment gave the ratio of throws with no-resistance bridge to those with condenser only to be $1{\cdot}26 \pm 0{\cdot}07$, the lengths of the line before and after the electrometer being 117·5 m. and 48 m. respectively. Applying now to this case the theory of the phenomena analogous to that developed in the early portion of this paper, and substituting for ρ^2 and s in the equation so derived the values 0·4776 and 0·000564 as shown in Table II., we obtain for the reflexion coefficient of the condensers alone

$$\rho' = 0{\cdot}82 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

This value, while differing but little from that of ρ for the condensers, spark-gap, and induction-coil at the oscillator, differs in the direction that one might anticipate. It also supports the value previously found for ρ, in that ρ' though greater than ρ is still distinctly less than unity. It will be seen from the end of the article in 'The Electrician,' April 9th, 1897, that Mr. Heaviside scarcely expected the long-wave theory to be valid for the high-frequency waves here used.

Conclusion.—The chief results arrived at may be thus summarized :—

1. The value of the attenuation constant σ for the line experimented upon and the waves in use is of the order 0·000013, where the amplitude of the waves is affected by the factor $e^{-\sigma x}$ in traversing x cm. of the line. The energy of the waves is thus reduced to half by traversing about 250 m. of the line.

2. The reflexion coefficient applicable to the primary oscillator and induction-coil as used to generate the waves is of the order 0·69, or rather more than half the energy is lost on reflexion. The reflexion coefficient of a precisely similar arrangement of condenser-plates but devoid of spark-gap and induction-coil is rather greater but still distinctly less than unity. On the long-wave theory the second of

these reflexion coefficients would be precisely equal to unity and the first of them practically so.

I have to thank Mr. H. A. Ratcliff for help at the beginning of this work, and especially Mr. R. S. Willows for his valuable assistance at the later experiments and their reduction.

University College, Nottingham,
May, 1897.

XXI. *Experiments on the Condensation and Critical Phenomena of some Substances and Mixtures.* *By* Dr. J. P. Kuenen *.

1. The investigation contained in this paper is a continuation of a research on "The Condensation and the Critical Phenomena of Mixtures of Ethane and Nitrous Oxide," which was read before the Physical Society some time ago †. The results arrived at in the latter paper were stated as follows:—

(1) All mixtures of C_2H_6 and N_2O which contain more than 0·1 of C_2H_6 have critical temperatures lying beneath the critical temperature of C_2H_6.

(2) The vapour-pressures of the mixtures lie partly above those of N_2O, and accordingly show a maximum situated near 0·2 of C_2H_6.

(3) This maximum does not disappear with rise of temperature, but remains up to the critical region; the maximum-curve reaches the plaitpoint-curve.

(4) The mixtures ranging between 0·2 and 0·5 of C_2H_6 have retrograde condensation of the second kind, all the others retrograde condensation of the first kind.

I have now discovered two more instances of mixtures with similar properties, viz. mixtures of ethane and acetylene, and mixtures of ethane and carbonic acid. I will begin by relating some experiments which were made in connexion with the former paper, and afterwards give an account of the results obtained with those mixtures.

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* Read June 25, 1897.

† Kuenen, Proc. Phys. Soc. vol. xiii. pp. 523–548.

A. *On Ethane prepared by different methods.*

2. The ethane used in my former experiments had been prepared from sodium acetate by electrolysis. Its vapour-pressures and critical constants, especially the latter, were different from those given by other observers. The different values are the following, *p* being expressed in atmospheres :—

	Dewar [*].	Olszewski [†].	Haenlen [‡].	Kuenen [§].
t_c . . .	35°	34°	34·5°	32·05°
p_c . . .	45·2	50·2	50	48·8

Ethane from Ethyl Iodide.

3. In order to ascertain the probable reason of this divergence I prepared some ethane from ethyl iodide, the method used by Haenlen, and investigated its properties. The results have been communicated to the Royal Society of Edinburgh. I will therefore simply state my conclusions here so far as they bear upon the point in question.

Ethane from ethyl iodide contains a variable quantity which may go up to several per cent. of an admixture of higher critical temperature and higher density than ethane, probably butane. This impurity raises both the critical temperature and pressure, and lowers its vapour-pressures. Pure ethane is difficult if not impossible to obtain by this method.

This result accounts for Haenlen's critical values being higher than mine (and probably also for Olszewski's figures), and it shows at the same time the greater purity of my ethane. Even this was not absolutely pure, as shown by some change of the vapour-pressure with volume. But whatever the admixture may be that this was due to, it was probably too small to affect the critical temperature by more than a few tenths of a degree and the critical pressure by more than some tenths of an atmosphere.

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[*] Phil. Mag. (5) xviii. p. 214.

[†] *Bulletin Ac. des Sciences de Cracovie,* 1889, p. 27.

[‡] Lieb. *Ann.* cclxxxii. p. 245.

[§] Proc. Phys. Soc. xiii. p. 533.

Ethane from Sodium Acetate.

4. In order to confirm my results, and at the same time to obtain some more pure ethane for further experiments, I repeated the preparation of ethane from sodium acetate. The anode, consisting of two pieces of platinum foil (joint surface 150 cm.^{2}), was placed inside a good-sized porous pot, care being taken to obtain a high current-density [*]. A current of about 15 amperes was used. Special precaution was required to prevent the indiarubber stopper which closed the porous pot from being burnt by the heat developed. A glass tube, passing through the stopper, was bent into a U-shape outside and contained some mercury. The contact with the platinum foil was made by platinum wires soldered to the foil.

The gas was washed in sodium hydrate, fuming sulphuric acid, and potash, and collected in a gasometer which was filled with a solution of potash. This gasometer consisted of two carboys, closed with indiarubber stoppers and connected by a wide glass tube. The yield of ethane was very high, very little oxygen being formed. The oxygen was moreover absorbed by introducing a piece of phosphorus into the gasometer.

5. In order to purify the gas further, it was compressed by means of a Natterer pump into a small copper cylinder fitted with high-pressure stopcocks [†]. In this process it was once more washed in potash and subsequently dried by calcium chloride and phosphorus pentoxide. Pump and cylinder were previously exhausted by a mercury air-pump. During the compression the copper cylinder was placed in solid carbonic acid. The pressure required to liquefy the gas was pretty high: this shows that it contained a considerable

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[*] Murray, Journal Chem. Soc. lxi. pp. 10–36.

[†] Most of my high-pressure apparatus are manufactured by Giltay, Delft, Holland, and are of the same type as those used in the Physical Laboratory at the University of Leyden (Director, Prof. Kamerlingh Onnes).

quantity of some permanent gas ; this may have been air which in some unknown manner had got into the gas. By thoroughly boiling the ethane, as much of this admixture was removed as possible. The results obtained with this ethane show that some admixture was left, of about the same amount as in the gas which I used in my former experiments. It is noteworthy that with other substances, such as carbonic acid and nitrous oxide, I always succeeded in removing practically all impurity by this process ; for some reason the same thing seems more difficult with ethane. It is not impossible that ethane forms a mixture of constant boiling-point with the admixture, whatever it may be.

6. The apparatus which was used for filling the compression-tubes with the gases consisted almost of one piece of glass. The mercury air-pump was sealed to the double mercury gasometer which I used for preparing the mixtures (see below). This was connected at the other side with a tube leading to the compression-tube, and another tube through which the purified gas was supplied from the compression cylinder. The connexion with the cylinder was made by means of a copper tube, sealed with sealing-wax into a glass tube, the connexion with the compression tube being made by a ground joint. This was so adjusted, that the tube after being filled could be turned upright without being detached, by which manipulation the tube was shut off at the bottom by some mercury. There are two reasons why I prefer this manner of filling the tubes to sealing them off. In the first place many gases are more or less decomposed at the temperature of softened glass, and in the second place the sealing place is apt to end in a sharp point, while for the proper action of the little iron stirring-rod which I use inside the tubes, a rounded top is absolutely necessary. It need hardly be added that the apparatus was fitted with a certain number of safety-tubes dipping in mercury. I found that the amount of spring in the apparatus was quite sufficient, and that special glass springs were superfluous.

7. For the observations properly speaking, I used Ducretet's apparatus. The temperatures of air-manometer and gas-tube were kept constant with the running water of the supply. A constant water-pressure was obtained by means of an overflow

arrangement, and the current before passing round the gas-tube was heated by a long coil in warm water. A sufficiently constant temperature was obtained in this manner.

TABLE I.

Ethane from Sodium Acetate (April 1897).

t.	p_b.	p_e.	v_b.	v_e.
5·55	26·90	27·42	231·2	33·0
14·5	32·91	33·52	177·1	35·6
15·0	33·23	33·77	176·1	35·5
23·05	39·74	40·27	130·9	39·0
30·2	46·35	46·72	94·5	46·2
32·2	48·64	C	61·5	

8. Table I. gives the results obtained: p_b and v_b are the pressures and volumes when the condensation begins, p_e and v_e when the whole substance is liquefied; p is expressed in parts of an atmosphere, the v's in this and in all the other tables are expressed in an arbitrary unit, but reduced to the some initial volume at 0° and 1 atmosphere. C indicates the critical point. The experimental determination of the p_e's gives no difficulty if the stirring-rod is continually used. For p_b this is not always the case, because stirring is not so effective when the volume is much larger. In cases of doubt p_b and v_b were therefore determined graphically as the points of intersection in the p-v diagram of the two parts of the isothermal, the part where the substance (mixture) is homogeneous and the part where there are two phases. Another advantage in drawing the latter part of the isothermal is that from its shape and the direction of its curvature some idea may often be obtained as to the nature of the admixture *.

9. Tables II. and III. contain values obtained for two samples of ethane on a former occasion. The pressures as shown by a comparison of the p-t curves are this time slightly

* Compare a paper on ethane in the Proc. R. S. of Edinburgh, 1897. I thought it unnecessay to publish the p-v diagrams as they offer no special interest.

Table II.

Ethane. (April 1895.) 1st sample.

t.	p_b.	p_e.
5·85	27·38	27·92
10·65	30·45	31·11
15·4	33·84	34·48
22·4	39·73	40·25
29·35	45·94	46·25
31·0	47·64	47·95
31·95	48·81 C	

Table III.

Ethane. (April 1895.) 2nd sample.

t.	p_b.	p_e.
3·95	26·17	26·89
15·4	34·11	34·66
22·4	39·89	40·32
29·3	45·89	46·19
32·05	48·91 C	

lower and the critical temperature a little higher. Absolutely pure ethane would probably have pressures slightly lower even and a slightly higher critical temperature. The p-t curves in the diagrams for ethane and the other single substances are drawn with p_b; as this value is probably nearer the true value of p than p_e.

B. *Mixtures of Carbonic Acid and Acetylene.*

10. In my former paper I gave a first instance of mixtures having critical temperatures below those for the component substances. The only instance of critical temperatures above those of the components seemed to be that of mixtures of carbonic acid and acetylene. According to an observation of Dewar's [*], a mixture of $\frac{1}{2}$ CO_2 and $\frac{1}{2}$ C_2H_2 has a critical temperature of 41° C., those for carbonic acid and acetylene being 31° and 37° respectively. It seemed necessary to try the experiment, because experiments on mixtures have always

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[*] Proc. R. S. of London, xxx. p. 542.

to be accepted with caution, unless it appears that exceptional precautions have been taken to avoid retardation and impurities*. The phenomenon, if confirmed, would have been worth completely investigating. The results which I have obtained contradict Dewar's experiment.

Preparation of Carbonic Acid.

11. Both carbonic acid and acetylene were prepared before I was in a position to apply the method of purification described for ethane by condensation and distillation. I had therefore to resort to chemical methods of purifying the gases. Carbonic acid was made from sodium bicarbonate and sulphuric acid. The acid was contained in a boiling-flask provided with a dropping-funnel with the sodium bicarbonate solution. All solutions used were previously boiled and cooled in a current of carbonic acid. The gas passed through sulphuric acid and phosphorus pentoxide. Before the solution was admitted the whole apparatus had been exhausted with a mercury air-pump.

12. The purity of the carbonic acid was first tested in the compression-apparatus and appeared to be quite satisfactory.

TABLE IV.

Carbonic Acid.

t.	p_b.	p_e.	Amagat.
14·95	50·02	50·24	50·0
20·5	57·16	......	57·0
25·85	64·92	65·06	64·5
31·1	73·26	C	

Amagat's values for the critical point are 31°·35, 72·9 atm. The agreement is very close, the deviations of my values

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[*] Prof. Dewar, on learning that I had repeated this experiment of his (forming part of an essentially qualitative investigation carried out in 1880), told me that his acetylene contained C_2H_3Br. This accounts for the high critical temperature and low vapour-pressures observed by him.

being probably due to a slight admixture of permanent gas. This would, in fact, make the vapour-pressures and critical pressure higher and the critical temperature lower. Similar deviations will be noticed by comparing some of Andrews's values with Amagat's, the former having been reduced to atmospheres.

t.	Andrews.	Amagat.
13·09	48·05–49·50	47·6
21·46	58·94–61·35	58·2

Andrews's critical constants are 30°·9 and about 73 atmospheres. The deviations are in the same direction, only much more pronounced. Andrews's carbonic acid contained about $\frac{1}{500}$ of air.

Preparation of Acetylene.

13. This gas was prepared in the same apparatus as carbonic acid, from calcium carbide and water. The water had been previously saturated with carbonic acid. The gas was washed in silver nitrate to absorb the phosphoretted hydrogen and sulphuretted hydrogen which it contained. It was freed from carbonic acid by soda-lime and dried in sulphuric acid and phosphorus pentoxide. Two samples were obtained. The first probably contained some air and also some phosphoretted hydrogen. In preparing the gas for the second time, a second washbottle with silver nitrate was introduced, and this bottle was only just beginning to get blackened, when the acetylene was collected. The only possible impurity of this sample is therefore a trace of air.

Some of this sample was used in making the mixture with CO_2, the results for which are communicated below. Some mixtures of the first sample with CO_2 gave, however, similar results. The results obtained with the purer sample are contained in Table V.

14. Evidently the gas contained an admixture of some other substance. If this admixture is air, the vapour-pressures are slightly higher than for pure acetylene, and the critical

temperature a few tenths of a degree too low. In this case also the p_b's were used in drawing the vapour-pressure curves in the diagrams.

TABLE V.

Acetylene.

t.	p_b.	p_e.
14·95	38·09	38·66
20·15	43·18	43·75
26·8	50·29	50·85
27·1	50·55	51·12
30·8	55·05	……
35·25	61·02 C	

The pressures agree very well with those given by Villard [*], who finds 37·9 at 15° C., and 42·8 at 20°·2. Ansdell's [†] values are a good deal different, his pressures being 32·77 at 13°·5, 39·76 at 20°·15, 48·99 at 27°·55, 56·20 at 31°·6. For the critical point he found 68·0 atmospheres and 36°·9 C. As to Pictet's [‡] values, they show no resemblance whatever with the others. His pressures were: 29·0 at 14°·1, 33·5 at 19°·5, 38·5 at 27°·6, and 48·0 at 36°·5 (critical point).

Mixture of Carbonic Acid and Acetylene.

15. The mixture was made in a mercury gasometer consisting of two parallel wide glass tubes of 300 c.c. each, provided at the bottom with a tap, an indiarubber tube, and a funnel with mercury sufficient to fill up the whole of the tubes. At the top the tubes ended in threeway stopcocks communicating together, the whole, as described before, connected on one side with the pump, and at the other side with the rest of the apparatus. The mixture contained $\frac{1}{2}C_2H_2$ and $\frac{1}{2}CO_2$.

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[*] *Ann. de Chimie et Physique,* March 1897.

[†] Proc. R. S. of London, xxix. p. 211 (1879).

[‡] *Zeitschrift Compr. und flüssige Gase,* i. p. 32 (1897).

16. The results obtained are the following :—

TABLE VI.

Mixture 0·5 $C_2H_2 - $0·5 CO_2.

t.	p_b.	p_e.
7·35	35·98	36·90
14·95	43·46	44·62
20·15	49·54	......
24·75	55·18	56·10
26·15	57·10	57·95
31·55	64·79	65·17
32·55	66·52 C	

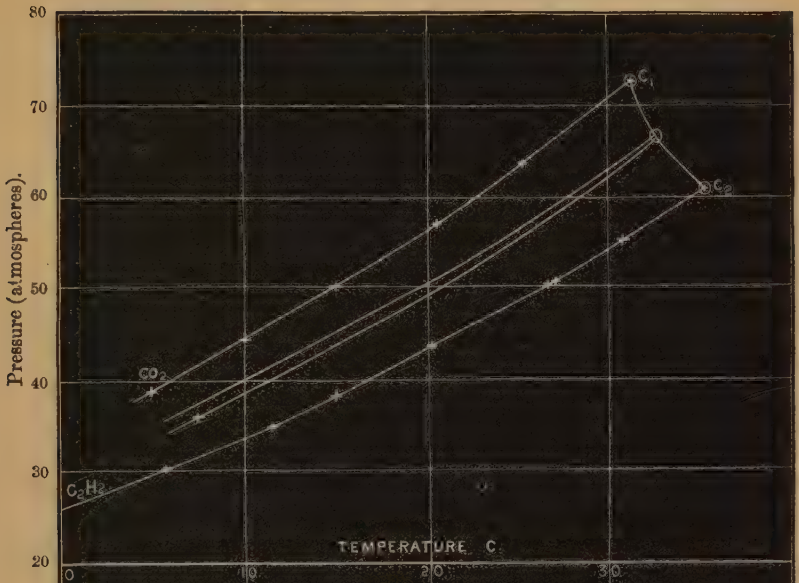
The differences between p_b and p_e (fig. 1), the pressures at the beginning of the condensation and at the end, are somewhat larger than for acetylene, as might be expected. It would be possible to make an allowance for the impurity of the component gases, and in that manner obtain approximate corrected values for the pressures. This would, however, be a rather uncertain calculation, and in drawing the border-curve for this mixture I have preferred to use the direct experimental values. Obviously, though the slight impurities of the CO_2, and specially of the C_2H_2, also affect the values for the mixture, the divergence from the true values can only be of the same order of magnitude as for the components.

17. The diagram (fig. 1) shows that the plaitpoint curve is a line with small curvature and that mixtures of carbonic acid and acetylene have critical temperatures between those for the pure gases. The border-curve for the mixture investigated is relatively narrow. This entails small differences in composition between the coexisting phases, the vapour containing only a little more carbonic acid than the liquid. It also follows that the critical phenomena will closely resemble those of a pure substance. This was found to be the case, the plaitpoint temperature and critical temperature nearly coinciding.

18. An instance of a critical temperature above those for the components has therefore not been discovered as yet. From the theory it may be deduced that this phenomenon

Fig. 1.

(*Vide* Tables IV., V., VI.)



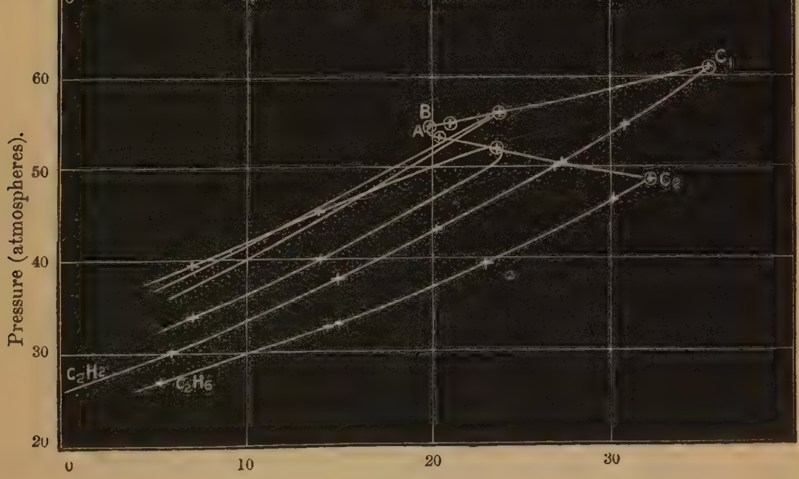


Fig. 2.

(*Vide* Tables I., V., VII.–XI.)

will probably occur for mixtures which have a minimum vapour-pressure at low temperature ; in the same way as the critical temperatures below those for the components seem to occur for mixtures which have a maximum vapour-pressure (as for nitrous oxide and ethane).

The law of the connexion between the two phenomena will be again referred to in connexion with the application of Prof. van der Waals's theory.

C. *Mixtures of Acetylene and Ethane.*

19. The relative position of the vapour-pressure curves and critical points in the *t-p* diagram of these gases was such as to promise interesting results for mixtures of them : in fact the same reasoning holds for this combination which led me originally to the choice of nitrous oxide and ethane. If we consider the curves for the four gases the critical temperatures of which are between 30° and 40°—CO_2, C_2H_6, C_2H_2, and N_2O—we find that besides with the combinations C_2H_6–N_2O and C_2H_6–C_2H_2 we may expect similar phenomena with C_2H_2–N_2O, a combination which I have not been able to try. Results obtained with the couple C_2H_6–CO_2 will be described later on. They agree with those of N_2O–C_2H_6, and C_2H_2–C_2H_6. The case C_2H_2–CO_2 was discussed above. The combination CO_2–N_2O offers no interest *a priori*. Moreover, some time ago I investigated a mixture of these two gases, but did not obtain any remarkable results.

Mixtures of some of these four gases with hydrochloric acid (51°) or methyl fluoride (44°·9) also promise new results. Mixtures of hydrochloric and carbonic acids were investigated by Dewar * and Ansdell †. Their figures differ from each other, but the plaitpoint curve does not seem to offer any peculiarities. It is hardly necessary to add, that a number of combinations of substances at other temperatures might be mentioned which would probably display similar phenomena.

20. I began by preparing two mixtures of acetylene and ethane containing 0·68 and 0·27 of the latter gas respectively.

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* Proc. R. S. of London, xxx. p. 543.

† *Ibid.* xxxiv. pp. 113–119.

After having got the constants for these two and not having any more acetylene available at the time, I made new mixtures by partly exhausting the compression-tube containing the 0·27 C_2H_6 mixture and introducing fresh quantities of ethane. In this manner four more mixtures were obtained. Though great care was taken not to introduce any traces of air into the tubes by this repeated process, it is not a safe method, and the results seem to show, that some impurity must have got into the mixture. Moreover, the calculation of the composition of each fresh mixture was again somewhat uncertain. The consequence is, that the last mixture prepared in this manner, which, according to the calculations, contained 0·76 C_2H_6, showed higher changes of vapour-pressure ($p_e - p_b$) than the first mixture ($x = 0·68$), and at the same time a slightly lower critical temperature and higher critical pressure. All this does not agree with the position and shape of the plaitpoint curve. The results for this mixture had therefore to be rejected. This also makes the values for the other three mixtures (especially the last one) doubtful. It must, however, be understood that this small uncertainty does not make the general result in the least doubtful, only the curves would be slightly different for the mixtures if pure.

TABLE VII.

Mixture 0·68 C_2H_6–0·32 C_2H_2.

t.	p_b.	p_e.	v_b.	v_e.
6·97	……	39·48	……	33·9
7·05	34·1	……	172·5	
13·95	40·35	45·25	134·3	37·2
22·2	49 17	51·83	88·2	46·5
23·05	……	52·35	……	50·2
23·15	……	52·40 } M	……	50·7 } M
23·35	……	52·41	……	54·3
23·4	……	52·32 P	……	57·4 P
23·45	51·31	……	72·9	……
23·5	51·33	……	72·9	……
23·55	……	52·29	……	59·1
23·6	……	52·22	……	60·2
23·65	51·72	52·16	67·2	60·3
23·75	52·06	C	65·1	

Table VIII.
Mixture 0·61 C_2H_6 — 0·39 C_2H_2.

t.	p_b.	p_e.	v_b.	v_e.
8·85	39·6	44·04	136·6	34·3
13·97	44·6	48·42	112·9	37·3
20·15	52·10	53·68 } M	74·6	48·4
20·55	52·61	53·68	70·6	52·4
20·75	53·06	53·47 P	67·5	59·5
20·77	53·4 ?	C	62·5	

Table IX.
Mixture 0·51 C_2H_6–0·49 C_2H_2.

t.	p_b.	p_e.	v_b.	v_e.
8·4	40·87	45·60	130·4	33·1
14·0	46·4	50·50	106·3	36·1
19·35	53·27	55·02 } M	73·2	46·1
19·75	54·10	55·02	65·2	51·2
19·85	……	54·87 P	……	54·4
19·9	54·24	……	64·4	……
19·95	54·57	54·72	60·7	57·7
19·96	54·65 ?	C	58·2	

Table X.
Mixture 0·41 C_2H_6–0·59 C_2H_2.

t.	p_b.	p_e.	v_b.	v_e.
7·95	……	42·70	……	31·4
8·0	39·95	……	136·2	……
14·0	46·1	48·19	107·3	34·6
19·35	52·0	53·67	83·1	40·6
20·95	54·50	55·11 } M	67 ?	47·9
21·0	54·62	55·12	……	50·0
21·05	……	55·12	……	50·6
21·1	……	55·05 P	……	53·7
21·12	……	55·02	……	55·2
21·15	54·99	C	55·1	

Table XI.
Mixture 0·27 C_2H_6–0·73 C_2H_2.

t.	p_b.	p_e.	v_b.	v_e.
7·6	38·25	39·61	148·4	29·7
13·95	44·63	45·64	114·5	31·7
22·2	53·9	54·55	75·7	39·0
23·75	56·05	56·27 } M	61·7	47·7
23·8	……	56·25 P	……	51·1
23·85	56·23	C	55·4	

21. The Tables VII.–XI. contain the results for the five mixtures; the diagram (fig. 2) shows the relative position of the vapour-pressure curves for the pure substances, the border-curves for the mixtures, and the plaitpoint curve which envelops the border-curves. Only two of the border-curves, for mixtures 0·68 and 0·27, have been drawn in order not to make the diagram too complicated.

The diagram is qualitatively the same as that for nitrous oxide and ethane: quantitatively there is a difference in the circumstance that the mixture of minimum critical temperature and the mixture of maximum vapour-pressure are much nearer to each other than in the other case. Mixture 0·51 was about the mixture of maximum pressure (point B), while the minimum critical temperature (point A) occurs for a mixture containing only a little more ethane ($x=0·53$ or $0·54$). The same points B and A were much farther apart for N_2O–C_2H_6, where they belonged to mixtures $x=0·2$ and $0·5$ respectively.

22. It was stated in my former paper that the critical phenomena of the mixtures of N_2O and C_2H_6 are little marked. Between C and A, and also between C and B, the condensation should be retrograde of the first kind between the two critical temperatures t_p and t_c; between A and B the condensation should be retrograde of the second kind. The latter phenomenon has never yet been observed owing to disturbing circumstances such as gravitation and small impurities. For C_2H_2–C_2H_6 the chances of observing special critical phenomena between A, B are obviously even smaller than for N_2O–C_2H_6.

I succeeded this time in observing critical phenomena of the first kind with all five mixtures. This is chiefly owing to the temperatures being more constant than formerly. The phenomena were very marked for the mixtures 0·68, 0·61, and 0·51, much less marked with mixtures 0·41 and 0·27. Though the relative position of P (plaitpoint) and C (critical point) is the same along the curves C_1B and C_2A, this is not the case as regards the point of maximum pressure M. Between B and C_1, M is between P and C; along AC_2, P is between M and C. The results with mixtures 0·68, 0·61, and 0·27 are in accordance with this rule; mixture 0·41 shows M to be near P but on the wrong side; mixture 0·51, which is about the mixture of maximum vapour-pressure, should have one critical point; evidently it contained some other admixture. Moreover, the differences between p_b and p_e are considerable for this mixture, while they should be very small, not larger than for the components used. Altogether the results show that at least some of the mixtures contained a small admixture of air. The figures are therefore less trustworthy than those for N_2O–C_2H_6 or those for CO_2–C_2H_6, the combination to be described presently.

The v–x curves have the same character as those for N_2O–C_2H_6: it seems therefore unnecessary to publish them.

D. *Mixtures of Carbonic Acid and Ethane.*

23. The ethane used in these experiments was the same as for the mixtures with acetylene. The carbonic acid was taken from an ordinary carbonic-acid cylinder, after having been distilled and purified in a small high-pressure cylinder in the way as described for ethane (see § 5).

TABLE XII.

Mixture 0·50 C_2H_6–0·50 CO_2.

t.	p_b.	p_e.	v_b.	v_e.
8·8	......	47·47	......	32·2
8·85	45·74	......	113·8	......
9·05	45·94	......	111·8	......
9·1	......	47·74	......	32·4
14·95	52·94	54·10	82·6	37·1
17·55	56·63	57·18 M	62·4	44·8
17·75	56·90	57·24 P	60·8	51·4
17·85	56·99	57·15	59·9	55·9
17 88	57·10		57·4	

TABLE XIII.

Mixture 0·43 C_2H_6–0·57 CO_2.

t.	p_b.	p_e.	v_b.	v_e.
8·95	47·27	48·46	106·8	30·7
14·95	54·50	55·30	77·4	35·5
17·55	......	58·42 M	......	44·8
17·58	58·18	.....	57·9	......
17·62	58·37	P and C	52·7	

TABLE XIV.

Mixture 0·30 C_2H_6–0·70 CO_2.

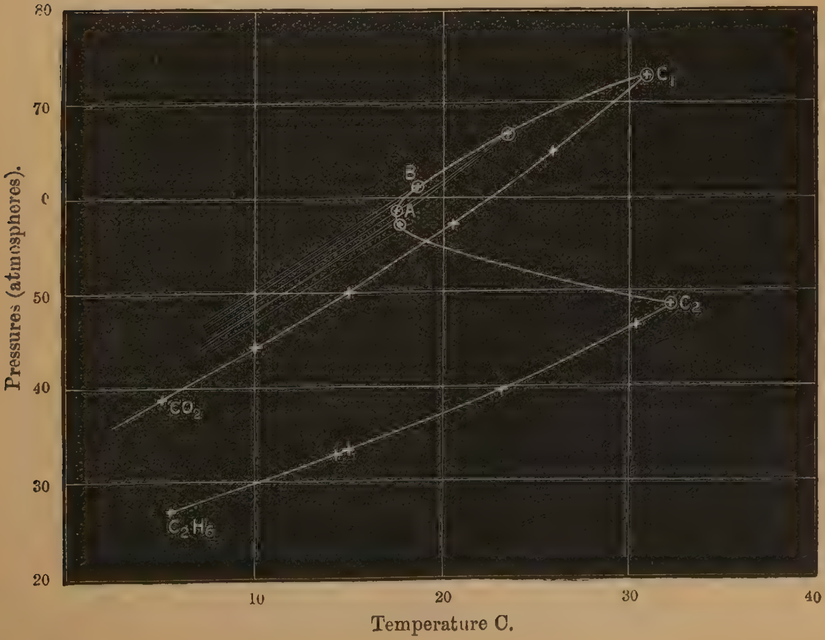
t.	p_b.	p_e.	v_b.	v_e.
8·95	48·58	49·07	106·0	28·8
14·95	55·82	56·28	79·8	32·2
17·28	59·02	59·31	66·9	35·0
18·68	......	61·19	......	41·4
18·69	61·14	......	53·3	......
18·73	......	61·28	......	42·8
18·75	61·21	61·31 M	52·5	46·1
18·77	......	61·29 P		
18·30	61·26	C	48·5	

TABLE XV.

Mixture 0·15 C_2H_6–0·85 CO_2.

t.	p_b.	p_e.	v_b.	v_e.
10·35	48·84	49·46	111·2	25·4
16·0	55·72	56·32	90	27·2
23·15	......	66·32	......	37·7
23·2	66·15	......	52·9	......
23·35	66·44	66·52 P	49·6	41·8
23·37	......	66·57 M	......	43·2
23·4	66·54	C	47·6	

Fig. 3. (*Vide* Tables I., IV., XII.–XV.)



24. These mixtures (Tables XII.–XV., fig. 3) show again the phenomena of minimum critical temperature and maximum vapour-pressure which had been found with mixtures of ethane with N_2O and C_2H_2. The minimum critical temperature occurs for a mixture $x=0{\cdot}45$ about (A), the maximum vapour-pressure for $x=0{\cdot}30$ approximately (B). These compositions are nearer together than for N_2O-C_2H_6, but not so near as for C_2H_2–C_2H_6.

25. Again, the critical phenomena characteristic of mixtures were very little marked. As just stated, mixtures containing ethane between the limits 0·30 and 0·45 ought theoretically to show the critical phenomena of the second kind. For mixture 0·43 points P and C were, however, so near together that I could not observe a difference. For mixture 0·50 P is between M and C; for mixture 0·30 M and P coincide, and very nearly also with C; for mixture 0·15 M seems to lie between P and C. All this agrees with the rules given above. The differences between p_b and p_e are very small, especially for the maximum mixture. This also is in accordance with theory. The results for this combination are more trustworthy than those for C_2H_2–C_2H_6, and as good or better than those for N_2O–C_2H_6.

The v-x diagram (not published) has the same character as for the two other combinations.

E. *Application of Van der Waals's Theory**.

26. If taken as generally as possible, van der Waals's theory of mixtures depends on the assumption of the existence of a relation between p, v, and t for a mixture of the same character as for pure substances. The p-v curve must be supposed to have an unstable part below a certain temperature, and in changing the composition of the mixture gradual changes in the relation between p, v, and t should take place. The laws of the condensation, the critical phenomena, in general of the coexistence of phases are then found by an application of thermodynamics in one of its many forms: van der Waals

* *Archives Néerl.* xxiv. pp. 1–56; *Zeitschrift physik. Chemie*, v. pp. 133–173; *Kon. Akad. van Wet. Amsterdam*, Nov. 24, 1894, pp. 133–137; *ibid.* Mei 25, 1895, pp. 20–30; Juni 29, 1895, pp. 1–12.

used the ψ function ($\psi=\epsilon-\mathrm{T}\eta$). (T = absolute temperature.) In fact wherever thermodynamics is used for purposes of this kind, the existence of a similar equation of condition must be presupposed, though the actual form of this equation can only be given in the simplest cases, and even then only qualitatively. It is well known that the theoretical equation for single substances given by van der Waals does not agree with the results of experiments, and the same is true for the analogous equation for mixtures of two substances which he uses in his theory. It is well, however, to bear in mind that a number of the results which he arrives at are independent of the form of the equation. Even those results which are not, still may have a resemblance to actual phenomena in the same manner as for single substances.

In the following account of some of the theoretical results I will only consider the case of the coexistence of vapour and liquid, *i. e.* a separation of the liquid into two will be excluded from the consideration.

I shall sometimes follow a method different from van der Waals's, chiefly for the sake of shortness and simplicity.

27. Let us suppose a set of isothermal curves (p-v diagram) to be drawn for the mixture as determined by observations of p, v, and t in homogeneous states, and completed by partly unstable interpolation-curves joining the curves in the homogeneous gaseous and liquid condition.

Suppose we forget for a moment that the substance is a mixture, and apply the Maxwell-Clausius criterion for finding the coexisting phases in the usual manner. We should then get the border or saturation-curve separating the homogeneous states from those where these are unstable or less stable than the separated states. The diagram thus obtained gives us the condensation-pressures, the vapour- and liquid-densities, and the critical point for the mixture, if supposed to behave like a pure substance, *i. e.* not dividing into phases which have a different composition. The real border-curve can only be found by a more complicated application of thermodynamics: still we may easily see one thing, that the real border-curve would lie entirely outside the hypothetical one just obtained. Instead of the horizontal line cutting off equal areas from the isothermal, we get a sloping line; this line has again to cut off equal areas. Obviously this is only possible by making it start lower and finish higher than the straight line. This is true at all temperatures, and the conclusion is, as stated, that the border-curve will be entirely outside the hypothetical one. From this it follows, *e. g.*, that the real critical temperature is higher than the critical temperature for the undivided mixture ; also that the critical isothermal does not touch the border-curve at the top M but at a point to the right C, where $\frac{dp}{dv} < 0$ instead of $=0$ as for a single substance. The volume in the critical point is obviously larger and the pressure smaller than the same quantities at the top of the border-curve.

28. The last fact about the critical pressure (point C) not being the maximum pressure (point M) for the mixture on the border-curve was pointed out above with reference to the *p-t* diagram. In fact the loops for the mixtures are nothing but the real border-curves represented in a different manner. The third point P which we considered in that case (plaitpoint), the point of contact of the loop and the enveloping curve (plaitpoint curve), cannot be found by a simple geometrical construction in the *p-v* diagrams. The connexion between the latter diagrams for different mixtures is not so simple as the connexion between the loops (border-curves) in the *p-t* diagram. A point of intersection of two loops gives the pressure and temperature at which the two mixtures to which the loops belong will coexist as vapour and liquid*. The enveloping-curve contains the points of intersection of loops infinitely near each other ; in other words, it gives the conditions of pressure and temperature of the coexistence of identical phases : at those points the mixtures are therefore in the critical condition, and will display the critical phenomenon. Evidently these points P do not in general coincide with either M or C. At the critical point C the mixtures do not show the critical phenomenon, the latter belonging to P, where the temperature is lower. This was fully explained on former occasions.

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* This is not so when a mixture of maximum or minimum pressure is between the two mixtures.

29. The properties of the v-x diagram were sufficiently discussed in my previous paper. Also its meaning as projection on the v-x plane of curves on the ψ surface studied by van der Waals. It was found in the first place that the maximum mixture in its critical point behaves entirely as a single substance; this fact would appear in the p-v diagram by the critical isothermal touching the border-curve at the top, and by points M, P, and C coinciding. If the maximum mixture is the same at all temperatures (as is approximately realized by N_2O–C_2H_6 within the limits of temperatures used), the real border-curve coincides entirely with the hypothetical border-curve examined above. If the maximum mixture changes with temperature the two curves coincide at the top only.

30. The consequence of this coincidence is that the critical point of the maximum mixture may be determined in exactly the same way as for a single substance: the conditions being

$$\frac{dp}{dv}=0, \quad \frac{d^2p}{dv^2}=0.$$

The direction of the vapour-pressure curve of the maximum mixture at B (*i. e.* the direction of the maximum-curve at that point) is also the same as for a single substance. This direction may be easily proved to satisfy the equation

$$\left(\frac{dp}{dt}\right)=\left(\frac{\partial p}{\partial t}\right)_v{}^*,$$

---

* This equation is true for single substances at the critical point, and for the maximum mixture at B; for other mixtures it would hold at the hypothetical critical point for the homogeneous mixture. The hypothetical vapour-pressure curve for a mixture lies in between the two branches of the loop in the p-t diagram, and comes to an end at this hypothetical critical point at some distance from the plaitpoint-curve. The equation consequently does not hold for a mixture at its plaitpoint. It seems therefore impossible to identify the expression $\left(\frac{\partial p}{\partial t}\right)_{vx}$ with the $\frac{dp}{dt}$ of the hypothetical vapour-pressure curve at the plaitpoint of an arbitrary mixture, although this is done by van der Waals in his interpretation of the equation which gives the difference of the value of $\left(\frac{dp}{dt}\right)_P$ and $\left(\frac{\partial p}{\partial t}\right)_{vx}$ (*Kon. Ak. Amsterdam*, Mei 25, 1895, pp. 13, 30; Juni 29, 1895, p. 3, form. (5)).

the index indicating the quantity which is kept constant in the differentiation. Van der Waals* has discovered that for single substances $\frac{T}{p}\frac{dp}{dt}$ is approximately equal to 7 at the critical point. It is an interesting confirmation of the theory that my maximum-curves also give a value about equal to 7 at point B.

31. An important question in connexion with this is, What is the direction of the plaitpoint-curve at B? In my former paper this point was left undecided. In the meantime Prof. van der Waals, to whom I showed my results, has published two papers† in which the properties of both points B and A are completely discussed. I will deduce here some of his results, sometimes using his equations in a different manner.

32. For $\frac{dp}{dt}$ of the plaitpoint-curve the following may be written:—

$$\left(\frac{dp}{dt}\right)_{P}=\left(\frac{\partial p}{\partial t}\right)_{vx}+\left(\frac{\partial p}{\partial v}\right)_{tx}\left(\frac{dv}{dt}\right)_{P}+\left(\frac{\partial p}{\partial x}\right)_{vt}\left(\frac{dx}{dt}\right)_{P}.$$

At B both $\frac{\partial p}{\partial v}$ and $\frac{\partial p}{\partial x}$ will be seen to be $=0$. This follows immediately from the conditions of the v-x diagram, $\frac{\partial p}{\partial v}$ being $=0$, because the line of equal pressure always touches the connodal curve at the plaitpoint, and is therefore parallel to the v-axis in this particular case; while $\frac{\partial p}{\partial x}=0$, because the point has still the properties of a maximum pressure-point. Neither $\frac{dx}{dt}$ nor $\frac{dv}{dt}$ can be ∞ along the plaitpoint-curve at B, because that would mean that the temperature went through a maximum or minimum. The result is that at B

$$\left(\frac{dp}{dt}\right)_{P}=\left(\frac{\partial p}{\partial t}\right)_{v}= \text{(see above)} \left(\frac{dp}{dt}\right) \text{ along maximum curve,}$$

i. e. the maximum curve touches the plaitpoint-curve at B.

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* "Continuity" &c., Supplementary Notes; *Kon. Ak. Amsterdam,* Mei 25, 1895, p. 27.

† *Kon. Ak. Amsterdam,* Mei 25, 1895, pp. 20–30; Juni 29, pp. 1–12.

The diagram in my former paper is therefore incorrect, as the plaitpoint-curve was drawn with a break at B.

33. As to point A, it may be proved that $\left(\frac{dp}{dt}\right)_{\mathrm{P}}$ is infinite there. We must use for this the conditions which hold for the point where a plait is split up. A plaitpoint is a point of the connodal curve on the ψ surface, but at the same time of the spinodal curve. The connodal curve is the curve traced out by the double tangent-plane: the spinodal curve, which is inside the connodal curve, separates the part of the surface where the substance is stable from the unstable part. The equation of this curve is

$$\mathrm{A}=\frac{\partial^2\psi}{\partial v^2}\frac{\partial^2\psi}{\partial x^2}-\left(\frac{\partial^2\psi}{\partial x\,\partial v}\right)^2=0.$$

Inside the curve $\mathrm{A} < 0$, outside $\mathrm{A} > 0$.

Applying the condition that $\mathrm{A}=0$ in the plaitpoint, we have

$$\left(\frac{d\mathrm{A}}{dt}\right)_{\mathrm{P}}=\left(\frac{\partial \mathrm{A}}{\partial t}\right)_{vx}+\left(\frac{\partial \mathrm{A}}{\partial x}\right)_{vt}\left(\frac{dx}{dt}\right)_{\mathrm{P}}+\left(\frac{\partial \mathrm{A}}{\partial v}\right)_{xt}\left(\frac{dv}{dt}\right)_{\mathrm{P}}=0.$$

Now at A $\frac{\partial \mathrm{A}}{\partial t}$ is > 0, because just before the division of the plait $\mathrm{A}<0$ in that point, and just after $\mathrm{A}>0$*. Moreover we have at point A $\left(\frac{\partial \mathrm{A}}{\partial x}\right)_{vt}=0$, and $\left(\frac{\partial \mathrm{A}}{\partial v}\right)_{xt}=0$. This will be seen to be true on considering that in the direction of the x-axis A passes from negative through 0 back to negative, in the direction of v from positive through 0 back to positive. The conclusion we come to is that either $\frac{dx}{dt}$ or $\frac{dv}{dt}$, or both, are infinite at A. The last alternative is the true one; but either of the three is sufficient for our purpose. If we put in the general equation for $\left(\frac{dp}{dt}\right)_{\mathrm{P}}$ in (32) $\left(\frac{dx}{dt}\right)_{\mathrm{P}}=\infty$, we find $\left(\frac{dp}{dt}\right)_{\mathrm{P}}=\infty$, because at A neither $\frac{\partial p}{\partial x}$ nor $\frac{\partial p}{\partial v}$ is 0, the direction of the p line not coinciding with either the v or the x axis.

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* A more rigorous proof may be given by working out $\frac{\partial \mathrm{A}}{\partial t}$: compare van der Waals, *l. c.* Juni 29, p. 2.

34. From $\left(\frac{dp}{dt}\right)_{\mathrm{P}}=\infty$ at A it follows, that the plaitpoint curve has a vertical tangent at A and the loop for the mixture of minimum critical temperature touches the plaitpoint curve at A in the ordinary manner. In my former paper I drew the plaitpoint curve with a break at A, as suggested by the experimental data, and drew some conclusions from this which now appear to have been erroneous. The two branches of the connodal curve at the moment of the division of the plait intersect as two straight lines. This was not sufficiently brought out in the diagram (fig. 11) *. Moreover the curves were drawn in such a manner, that a line parallel to the v-axis had, under certain circumstances, four points of intersection. This was an unnecessary assumption which, in cases of splitting up of the first plait, is not a probable one. Altogether the paragraphs 40–42 with the figures 7, 11, and 12 should be withdrawn.

35. All these results are entirely independent of the form of the equation between p, v, and T for the mixtures. We will now apply van der Waals's equation for some calculations regarding the mixtures investigated.

The equation between p, v, and T has the form

$$\left(p+\frac{a_x}{v^2}\right)(v-b_x)=\mathrm{RT},$$

in which a_x and b_x depend on the composition of the mixture x in the following manner :—

$$a_x=a_1(1-x)^2+2a_{12}x(1-x)+a_2x^2,$$
$$b_x=b_1(1-x)^2+2b_{12}x(1-x)+b_2x^2\ \dagger\ ;$$

a_1, b_1 being the constants for the first and a_2, b_2 for the second substance ; a_{12} is the attraction-constant between the two substances, b_{12} the volume-constant for the combination. According to the simple suppositions underlying the deduction of the formula, we should have :

$$\sqrt[3]{b_{12}}=\tfrac{1}{2}\{\sqrt[3]{b_1}+\sqrt[3]{b_2}\}.$$

36. Now the three combinations investigated offer a good opportunity for calculating a_{12} and b_{12}. We saw that the

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* Kuenen, Proc. Phys. Soc. xiii. p. 545.

† Lorentz, Wied. *Ann.* xii. p. 134.

same formulæ hold for the critical point of the max. mixture as for a single substance. From the T and p in the maximum point B we can therefore calculate a_x and b_x and deduce from these the a_{12} and b_{12}. The values of a_{12} will give some idea of the relative attraction of the two kinds of molecules, probably quite as accurate an idea as the ordinary a gives of the mutual attraction of the same molecules. The b_{12} obtained may be compared with the theoretical value.

37. The results of my calculations are the following :—

TABLE XVI.

	t_c.	p_c.	a.	b.
N_2O	309	71·9	0·00753	0·00197
C_2H_2	308·25	61·02	0·00881	0·00231
CO_2	304·1	73·26	0·00714	0·00190
C_2H_6	305	48·8	0·01078	0·00286
			a_{12}.	b_{12}.
N_2O–C_2H_6 ($x=0·18$) ...	302·8	65·3	0·00859	0·00239
C_2H_2–C_2H_6 ($x=0·51$) ...	292·96	54·7	0·00789	0·002295
CO_2–C_2H_6 ($x=0·30$) ...	291·8	61·3	0·008076	0·002357
CO_2–C_2H_2 ($x=0·50$) ...	305·55	66·5	0·00792	0·00210

I have added to the table calculated values of a_{12} and b_{12} for the combination CO_2–C_2H_2. The mixture of these two gases was proved to have a very narrow border-curve, and consequently the critical point observed cannot have been very much above the hypothetical critical point. The calculation of a_{12} and b_{12} from the critical constants, though not legitimate, must have given values for a_{12} and b_{12} not very far wrong.

The values of b_{12} calculated from the formula in § 35 coincide with the values in Table XVI. in the case of N_2O–C_2H_6 and of CO_2–C_2H_6 : for C_2H_2–C_2H_6 b_{12} is even smaller than b for C_2H_2. a_{12} is in all cases smaller than the mean of the a's, and for C_2H_2–C_2H_6 even smaller than a for C_2H_2. It is chiefly on this smallness of a_{12} that the existence of a minimum critical temperature depends.

38. Knowing a_{12} and b_{12}, we may calculate the critical

constants for the other mixtures, if supposed to remain undivided. It is with the temperature that we are chiefly concerned.

TABLE XVII.

x.	$273\frac{8}{27}\frac{a}{b}$.	T_c.
N_2O-C_2H_6.		
0·18	302·8	302·8
0·43	299·0	299·05
0·55	298·7	299·05
0·76	300·5	300·85
C_2H_2–C_2H_6.		
0·27	296·7	296·8
0·41	293·1	294·15
0·51	293·0	293·0
0·61	293·6	293·8
0·68	294·8	296·75
CO_2–C_2H_6.		
0·15	296·1	296·4
0·30	291·8	291·8
0·43	290·7	290·62
0·50	290·8	290·88

Table XVII. gives the calculated values of the hypothetical critical temperature as compared with the real critical temperature. The former is found to be everywhere lower than the latter (except for mixture 0·43 CO_2–C_2H_6, where the calculated value is 0°·08 above the observed T_c, a difference almost within the limits of experimental errors) : for the maximum mixture the two naturally coincide. The differences are very small, especially with N_2O–C_2H_6 and CO_2–C_2H_6. For C_2H_2–C_2H_6 they are somewhat higher for mixtures containing more C_2H_6 : this is in agreement with the fact that the critical phenomena were more marked for those mixtures than for the others. In plotting the values of Table XVII. against the x's, two curves are obtained, connecting the same points C_1 and C_2 and touching at a point corresponding to B. The curves are close together and the minimum of a/b

should belong to approximately the same value of x as the minimum T_c. The existence of a minimum for a/b depends on a_{12}/b_{12} being smaller than both a_1/b_1 and a_2/b_2[*]. Table XVIII. gives the values of x for the minimum of a/b and for the observed minimum critical temperature (point A). The two sets agree very closely.

TABLE XVIII.

	Min. a/b.	Min. T_c. : A.	Max. pressure.	
			Calculated.	Observed : B.
N_2O–C_2H_6 ...	$x=0{\cdot}52$	0·50	0·01	0·20
C_2H_2–C_2H_6 ...	$x=0{\cdot}50$	0·53	0·36	0·51
CO_2–C_2H_6 ...	$x=0{\cdot}48$	0·45	0·10	0·30

39. In Table XVIII. are also given the mixtures of maximum pressure as observed and as calculated from the theory. The latter calculation is simple. In coexisting phases we have the three conditions that $\frac{\partial\psi}{\partial v}, \frac{\partial\psi}{\partial x}$, and $\psi - v\frac{\partial\psi}{\partial v} - x\frac{\partial\psi}{\partial x}$ are equal. Now ψ may be found from the equation $p\ v\ T$, because $\frac{\partial\psi}{\partial v} = -p$. This leads to

$$\psi = -RTl(v-b_x) - \frac{a_x}{v} + \phi(x).$$

The two first conditions mentioned lead immediately to

$$\left(\frac{1}{v_1} + \frac{1}{v_2}\right)\frac{\partial b_x}{\partial x} = \frac{1}{a_x}\frac{\partial a_x}{\partial x},$$ [†]

remembering that x is the same in the two phases ; v_1 and v_2 are the volumes of the coexisting phases. At the critical point : $v_1 = v_2 = 3b_x$, and the equation becomes

$$\frac{2}{3b}\frac{\partial b_x}{\partial x} = \frac{1}{a_x}\frac{a_x}{\partial x}.$$

There is not a very good agreement between the values calculated from this formula and those observed. The points of agreement are that both sets of x's are smaller than the x's for the min. cr. temp., and that the smaller the x observed, the smaller also is the x calculated.

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[*] Van der Waals, *Arch. Néerl.* xxiv. p. 23.

[†] Van der Waals, *l. c.* p. 23.

Considering how insufficient van der Waals's formula is, we should not expect a better agreement; still the equation seems to give correct indications as regards the relative position of the two points A and B. This connexion may be better shown in the following manner: the condition for the min. of a_x/b_x may be written thus:—

$$\frac{1}{a_x}\frac{\partial a_x}{\partial x} - \frac{1}{b}\frac{\partial b_x}{\partial x} = 0,$$

and the condition for point B

$$\frac{1}{a_x}\frac{\partial a_x}{\partial x} - \frac{2}{3b_x}\frac{\partial b_x}{\partial x} = 0.$$

When $\frac{\partial b_x}{\partial x} > 0$ (in which case also $\frac{\partial a_x}{\partial x} > 0$), as is the case in the three combinations investigated, the first equation gives a higher value of x than the second *.

40. But another question arises—What happens to the maximum at very low temperatures? This can be found from the same equation given above (in 39) by putting v_1 (volume of the liquid) equal to b_x and neglecting $1/v_2$. The result is the same equation as holds for the min. cr. temperature. Probably therefore, if the equation does not mislead us here too much, approximately the same mixture which has a min. cr. temp. will be the mixture of maximum vapour-pressure at low temperatures. We may therefore expect the mixtures of C_2H_6 with N_2O, C_2H_2, and CO_2 which contain about 0·5 of each to have a max. vapour-pressure at low temperatures. By raising the temperature this maximum moves towards the substance of higher vapour-pressures (N_2O, C_2H_2, and CO_2). As we saw for C_2H_2–C_2H_6, this motion has been very small, the points B and A almost coinciding. It is not impossible that for other mixtures which have a max. pressure at low temperature the maximum reaches one of the components at higher temperature and thus disappears. Those

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* Van der Waals, *Kon. Ak.* 29 June, 1895, pp. 11–12.

mixtures would still have a minimum for a_x/b_x and, if the real critical temperatures are not too much above the hypothetical ones, they would also show a minimum critical temperature.

Instances of maximum vapour-pressures are:—ethyl alcohol and carbon disulphide (Berthelot[*]), chloroform and carbon bisulphide (Guthrie[†]), ether and carbon disulphide (Guthrie), propyl alcohol and water (Konowalow[‡]). Some of these combinations would be worth investigating at higher temperatures up to the critical condition.

41. Similar considerations hold for maximum critical temperature and minimum vapour-pressure. A minimum vapour-pressure at low temperature means, according to the equation, a maximum for a/b; and a maximum of a/b is the same as a maximum of the hypothetical critical temperature, and therefore *a fortiori* of the *real* critical temperature. The minimum pressure will move towards the substance of lower vapour-pressure, with rise of temperature, and may either disappear or else exist up to the critical region. It is easy to understand what the p-t diagram would be like in this case. But the case has not been observed yet. A minimum vapour-pressure was found by Roscoe and Dittmar[§] for hydrochloric acid and water and for nitric acid and water, by Konowalow[‖] for formic acid and water, by Friedel[¶] for hydrochloric acid and methyl ether. I hope to have an early opportunity of trying the latter combination, which will not require temperatures as high as the combinations with water.

42. One remark may be added. In the condensation of some substances it is assumed that an association takes place of molecules to more complicated systems. Substances with which that happens do not obey the laws of corresponding states, and van der Waals's formula would be even more inadequate in those cases than otherwise. We may expect similar phenomena to occur in the liquefaction of some

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[*] Berthelot, *Comptes Rendus*, lxvii. p. 430.

[†] Guthrie, Phil. Mag. (5) xviii. p. 513.

[‡] Konowalow, Wied. *Ann.* xiv. p. 34.

[§] Roscoe & Dittmar, Lieb. *Ann.* cxii. p. 327, cxvi. p. 327.

[‖] Konowalow, Wied. *Ann.* xiv. p. 34.

[¶] Friedel, *Comptes Rendus*, lxxxi. p. 152.

mixtures. An association of that kind one would feel inclined to identify with what is called the formation of molecular compounds. This would naturally have a great influence on the vapour-pressure, and it is not unlikely, that rules like those given above and deduced from van der Waals's formula would fail entirely in those cases. But these considerations can only be tested by experiment.

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Discussion.

Dr. S. P. Thompson asked whether diagrams characteristic of cyanogen had been obtained. Its remarkable polymerism suggested an interesting case for critical phenomena.

Dr. Kuenen thought such a substance might be worth investigating.

XXII. *On the Steady Motion of an Electrified Ellipsoid.*

By G. F. C. Searle, *M.A., Demonstrator in Experimental Physics, Cavendish Laboratory, Cambridge*[*].

At the Meeting of the Royal Society on 19th March, 1896, I read a paper on "Problems in Electric Convection." The first part of the paper is printed in the 'Philosophical Transactions of the Royal Society'[†], and contains the principles which are required in the solution of any problems about moving charges. The second part of the paper, which deals with the motion of a charged ellipsoid, was not published by the Royal Society. A few of the results are, however, stated in an abstract published in the 'Proceedings'[‡]. By the permission of the Royal Society I now publish my results for a moving ellipsoid. As frequent reference to my paper in the 'Philosophical Transactions' will be necessary, I shall use the notation {§ 5} and {(9)} to indicate the paragraph or equation referred to in that paper.

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[*] Read June 25, 1897.

[†] Phil. Trans. vol. 187 (1896) A. pp. 675–713.

[‡] Proc. Roy. Soc. vol. lix. p. 343.

When any system of electric charges moves with uniform velocity through the æther, the electromagnetic field, when referred to axes moving forwards with the charges, can be completely defined by means of a quantity Ψ, as was first shown by Prof. J. J. Thomson *. The electric force **E** and the magnetic force **H** are simple functions of Ψ. But besides **E** and **H** there is another vector of great importance, viz. the mechanical force **F** experienced by a unit charge moving with the rest of the system. The value of **F** I have shown {§ 10} to be given by the vector equation

$$\mathbf{F}=\mathbf{E}+\mu \mathrm{V}\mathbf{uH}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The equations of the field are {§ 4}

$$\operatorname{curl}\mathbf{F}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\mathbf{H}=\mathrm{K}\mathrm{V}\mathbf{uE}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If $v=\dfrac{1}{\sqrt{\mathrm{K}\mu}}$ is the velocity of light, and if α stand for $1-\dfrac{u^2}{v^2}$, then when the motion takes place parallel to the axis of x, we have {§ 4}

$$\mathrm{F}_1=-\frac{d\Psi}{dx} \qquad \mathrm{F}_2=-\frac{d\Psi}{dy} \qquad \mathrm{F}_3=-\frac{d\Psi}{dz}, \quad . \quad . \quad (4)$$

$$\mathrm{E}_1=-\frac{d\Psi}{dx} \qquad \mathrm{E}_2=-\frac{1}{\alpha}\frac{d\Psi}{dy} \qquad \mathrm{E}_3=-\frac{1}{\alpha}\frac{d\Psi}{dz}, \quad . \quad (5)$$

$$\mathrm{H}_1=0 \qquad \mathrm{H}_2=\frac{\mathrm{K}u}{\alpha}\frac{d\Psi}{dz} \qquad \mathrm{H}_3=-\frac{\mathrm{K}u}{\alpha}\frac{d\Psi}{dy}. \quad . \quad (6)$$

From these equations, since **E** has no divergence,

$$\alpha\frac{d^2\Psi}{dx^2}+\frac{d^2\Psi}{dy^2}+\frac{d^2\Psi}{dz^2}=0. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Here, and throughout the paper, the axes are supposed to move forward with the same velocity as the electrical charges.

Prof. W. B. Morton has considered the motion of an ellipsoid in a paper read before the Physical Society on 27th March, 1896 †. He obtains the two following results, viz.: (1) that the distribution of electricity is the same as if the ellipsoid is at rest, and (2) the value of Ψ when the ellipsoid moves along one of its axes.

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* Phil. Mag. July 1889.

† Proc. Phys. Soc. No. 71, August 1896, p. 180; Phil. Mag. xli. p. 488.

Prof. Morton obtains his result by the assumption first made by Mr. Oliver Heaviside, F.R.S. *, that a distribution of electricity on the surface of a charged body such as to give zero disturbance at all points inside the surface is an equilibrium distribution. Since **F** satisfies curl **F**$=0$ and **F** vanishes inside the surface, it follows that on the outside of the surface **F** is perpendicular to the surface. This implies that Ψ is constant over the surface. But as neither the electric force **E** nor the mechanical force experienced by each part of the charged surface (calculated from the Maxwell stress) is normal to the surface, I felt unable to accept the validity of Mr. Heaviside's assumption until I discovered {§ 15} that **F** is the mechanical force on an isolated moving unit charge, and that the term $-$V**GD**, which appears in the expression for the force experienced by the surface, has no influence in causing convection of electricity from one part of the surface to another. Here **D** is the electric displacement, and **G** the "magnetic current" $\mu\frac{d\mathbf{H}}{dt}$.

Since Ψ is a true potential for the mechanical force **F**, I have called Ψ the "electric convection potential."

When there has been established the boundary condition that Ψ is constant over the surface, with its consequence that there is zero disturbance within the surface, it is very easy to show that the distribution on an ellipsoid is the same for motion as for rest. Suppose the ellipsoid to have the same distribution as when it is at rest, so that $\sigma=qp/4\pi abc$, where q is the charge, a, b, c the axes of the ellipsoid, and p the perpendicular from the centre upon the tangent-plane at the point. Through any internal point M as vertex draw a slender double cone intercepting two areas N, N′ on the surface. Now the electric force due to a moving point-charge is still radial and still varies inversely as the square of the distance, although it alters with change of direction of the radius vector. Thus it follows just as in electrostatics, since $\sigma \propto p$, that the effects at M of N and N′ are exactly equal and opposite. The whole surface can be treated in the same manner, and thus it follows that **E**$=0$ at all internal points. Hence **H**$=0$ also. Thus the assumed distribution is in equilibrium and is therefore the actual distribution. Thus the motion has no influence upon the distribution, and this result is true whatever the direction of motion with respect to the axes of the ellipsoid.

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* 'Electrical Papers,' vol. ii. p. 514.

In order to find the state of the field near a charged ellipsoid moving with velocity u parallel to the axis of x, it is necessary to find a value of Ψ which shall be constant over the surface of the ellipsoid, shall vanish at infinity, and shall satisfy (7). We see at once that if $f(x, y, z)$ satisfies $\nabla^2 f=0$, then $f(x/\sqrt{\alpha}, y, z)$ satisfies (7). Now from electrostatics we know that

$$\Phi=\int_{\lambda}^{\infty}\frac{d\lambda}{\sqrt{(a'^2+\lambda)(b^2+\lambda)(c^2+\lambda)}},$$

where λ is connected with x, y, z by the relation

$$\frac{x^2}{a'^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1,$$

satisfies $\nabla^2\Phi=0$.

Hence

$$\Psi=\int_{\lambda}^{\infty}\frac{\mathrm{A}\,d\lambda}{\sqrt{(a'^2+\lambda)(b^2+\lambda)(c^2+\lambda)}}, \quad . \quad . \quad . \quad (8)$$

where λ is connected with x, y, z by the relation

$$\frac{x^2}{(a'^2+\lambda)}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1, \quad . \quad . \quad . \quad (9)$$

satisfies (7).

Writing a^2 for $\alpha a'^2$, (8) and (9) become

$$\Psi=\int_{\lambda}^{\infty}\frac{\mathrm{A}\,d\lambda}{\sqrt{(a^2+\alpha\lambda)(b^2+\lambda)(c^2+\lambda)}}; \quad . \quad . \quad (10)$$

$$\frac{x^2}{a^2+\alpha\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1. \quad . \quad . \quad . \quad (11)$$

This value of Ψ is constant over the surface of the ellipsoid a, b, c, for $\lambda=0$ at all points of this surface; it also vanishes at infinity, and it satisfies (7). It is therefore the value of Ψ required. To find the constant A we make σ have its proper value $q/4\pi bc$ at the end of axis a.

Now

$$\sigma = \frac{K}{4\pi} E_n = \frac{K}{4\pi} E_1$$

at the end of the axis.

But by (5)
$$E_1 = -\frac{d\Psi}{dx}.$$

Again, at $x=a$, $y=z=0$ we have $d\lambda/dx = 2a/\alpha$ and consequently

$$\frac{d\Psi}{dx} = \frac{d\Psi}{d\lambda}\frac{d\lambda}{dx} = -\frac{A}{abc}\cdot\frac{2a}{\alpha}.$$

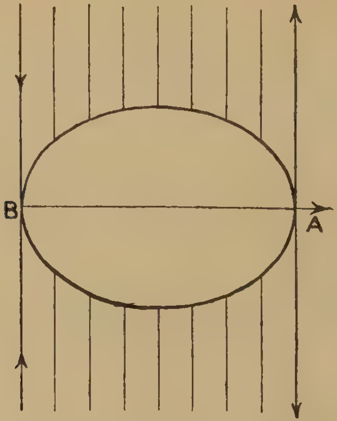
Hence
$$A = \frac{q\alpha}{2K}.$$

Thus, as Prof. Morton has also shown by the same method,

$$\Psi = \int_\lambda^\infty \frac{q\alpha \, d\lambda}{2K\sqrt{(a^2+\alpha\lambda)(b^2+\lambda)(c^2+\lambda)}}. \quad . \quad . \quad (12)$$

Now I have shown {§ 21} that if there is a surface A carrying a charge q, and any surface B is found for which Ψ is constant, then a charge q placed upon B and allowed to acquire an equilibrium distribution will produce at all points not inside B the same effect as the charged surface A.

Fig. 1.



Hence the ellipsoid (11) when carrying a charge q produces at all points not inside itself exactly the same disturbance as the ellipsoid a, b, c with the same charge.

If we make $a=b=c=0$, the surfaces of equal "convection potential" are the ellipsoids given by

$$\frac{x^2}{\alpha}+y^2+z^2=\lambda.$$

They are therefore all similar to each other. Thus the ellipsoid of this form produces exactly the same effect as a point-charge at its centre, and thus an ellipsoid of this form takes the place of the sphere in electrostatics. An ellipsoid with its axes in the ratios $\sqrt{\alpha}:1:1$ I have called a Heaviside Ellipsoid, since Mr. Heaviside* was the first to draw attention to its importance in the theory of moving charges. Whatever be the ratios $a:b:c$, the equipotential surfaces approximate to Heaviside ellipsoids as λ is made very great. The value of Ψ at the surface λ is $\frac{q\sqrt{\alpha}}{\sqrt{\lambda}}$.

Putting $c=b$ so that we have an ellipsoid of revolution, the axis of revolution being the axis of x, we see by taking $\lambda=-b^2$ that a uniformly-charged line of length $2\sqrt{a^2-b^2\alpha}$ lying along the axis of x produces exactly the same effect as the ellipsoid a, b, b. It may therefore be called its "image." When $b=a$ this length becomes $2au/v$. Thus, when a charged sphere is at rest it produces the same effect as a point-charge at its centre. When the sphere is in motion it produces the same effect as a uniformly-charged line whose length bears to the diameter of the sphere the same ratio as the velocity of the sphere bears to the velocity of light. When $u=v$, so that the sphere moves with the velocity of light, the line becomes the diameter of the sphere; and the same is true for an ellipsoid. Since when $u=v$ each element of the charged line produces a disturbance which is confined to the plane through the element perpendicular to the direction of motion {(46)}, it follows that the disturbance is entirely confined between the planes $x=\pm a$. Between them the electric force is radial to the axis of x and has exactly the same value, viz. $q/aK\rho$, as if the line had been of infinite length and had had the same line-density $q/2a$. Here ρ stands for $\{y^2+z^2\}^{\frac{1}{2}}$. The magnetic force is by (3) $qu/a\rho$. Hence the field between the planes $x=\pm a$ is independent of x. There are therefore no displacement-currents except in the two bounding-planes. There is an outward radial current in the front plane and an inward current in the back plane, the total amount of current in each case being qu, equal in amount to the convection-current carried by the ellipsoid.

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* 'Electrical Papers,' vol. ii. p. 514.

It appears, however, that at the velocity of light *any* distribution on *any* surface is in equilibrium. For the value of Ψ at any point near a moving point-charge is $\{(43)\}$

$$\Psi = \frac{q\sqrt{\alpha}}{\mathrm{K}\sqrt{x^2/\alpha+y^2+z^2}},$$

and this vanishes when $u=v$ (so that $\alpha=0$), even when $x=0$. Thus the value of Ψ for a point-charge vanishes, and the value of Ψ for *any* distribution being derivable from that for a point-charge by integration, it follows that Ψ has the constant value zero everywhere. Hence the charge is in equilibrium however it may be distributed. The same result follows from the expression {§ 19} for the force between two moving charges. When they move parallel to each other with the speed of light the force between them vanishes.

If the ellipsoid is more oblate than Heaviside's the limiting internal surface of ellipsoidal form, whose action is the same as that of the ellipsoid, is a disk of radius $\sqrt{b^2-a^2/\alpha}$, the axis of the disk coinciding with the axis of x.

The form of the lines of the electric force **E** due to an ellipsoid of revolution is easily found. Putting ρ^2 for y^2+z^2, the equilibrium surfaces are given by

$$\frac{x^2}{a^2+\alpha\lambda}+\frac{\rho^2}{b^2+\lambda}=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Now the mechanical force **F** is normal to this surface, and therefore

$$\frac{\mathrm{F}_\rho}{\mathrm{F}_1}=\frac{\rho(a^2+\alpha\lambda)}{x(b^2+\lambda)},$$

where $$\mathrm{F}_\rho{}^2=\mathrm{F}_2{}^2+\mathrm{F}_3{}^2.$$

But by (5),

$$\mathrm{E}_1=\mathrm{F}_1 \quad \text{and} \quad \mathrm{E}_\rho=\mathrm{F}_\rho/\alpha\,;$$

so that

$$\frac{E_\rho}{E_1}=\frac{1}{\alpha}\frac{\rho(a^2+\alpha\lambda)}{x(b^2+\lambda)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Now consider the conic

$$\frac{x^2}{a^2+\alpha\nu}+\frac{\rho^2}{b^2+\nu}=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The tangent of the angle which the geometrical tangent makes with the axis of x is

$$-\frac{x(b^2+\nu)}{\rho(a^2+\alpha\nu)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

But if the point x, ρ lies on both (13) and (15), it follows that

$$-\frac{x(b^2+\nu)}{\rho(a^2+\alpha\nu)}=\frac{1}{\alpha}\frac{\rho(a^2+\alpha\lambda)}{x(b^2+\lambda)}.$$

Hence by (14) and (16) the electric force is always tangential to the conic (15). But this conic has exactly the same equation as the equilibrium surfaces. Thus the single equation (13) represents both the equilibrium surfaces and the lines of electric force.

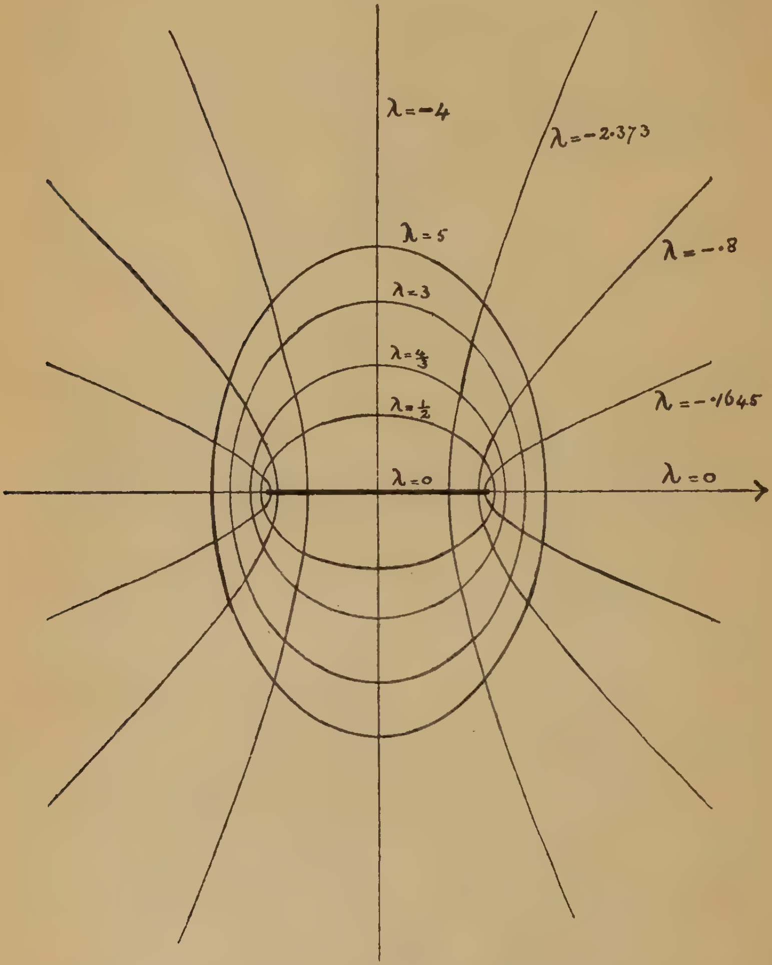
If any point x, ρ be taken, there are two values of λ which will satisfy (13) considered as a quadratic in λ. One value corresponds to an ellipsoidal equilibrium surface; the other to a hyperbolic surface whose lines of intersection with planes passing through the axis of x are the lines of electric force. The lines of electric force for a charged sphere in motion are not radial but form a series of hyperbolas.

Figs. 2 and 3 show the forms of the equilibrium surfaces and of the lines of electric force, for a line and a disk respectively, when $\alpha=\frac{1}{4}$ so that $u/v=\cdot 866$. In fig. 2 the curve marked $\lambda=\frac{4}{3}$ is a circle, a section of the sphere of which the line marked $\lambda=0$ is the "image." The semi-length of the line and the radius of the disk are each taken as unity.

I have attempted to find the lines of the mechanical force **F** these being everywhere perpendicular to the equilibrium surfaces. But the process involved an impracticable integration, and thus led to no result.

I will now write down the values of E and H at any point near the ellipsoid of revolution with axes a, b, b. Instead of λ it will be convenient to take as the parameter of any one of

Fig. 2.



the equilibrium surfaces its x axis and to denote this by h. Thus

$$h^2 = a^2 + \alpha\lambda\,;$$

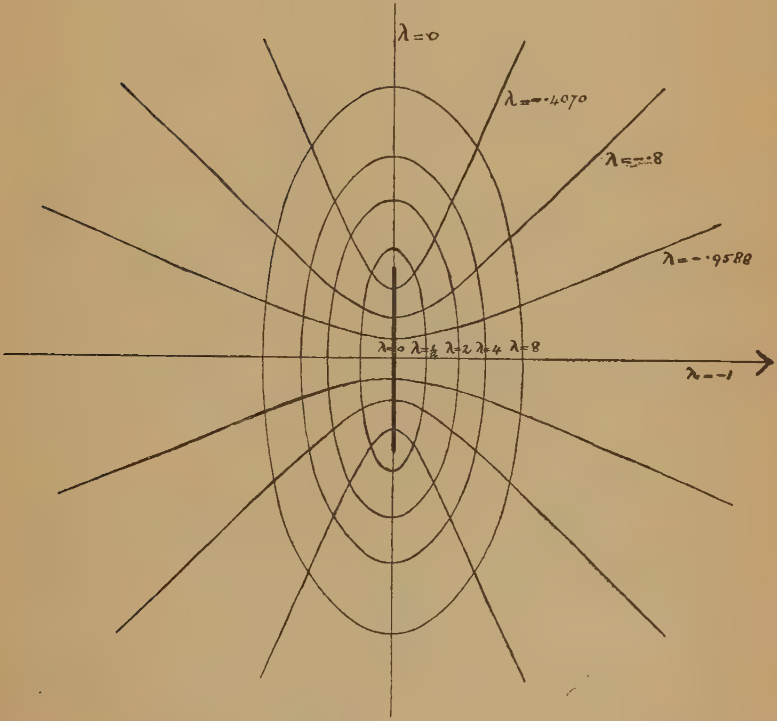
and consequently if we put l^2 for $a^2 - \alpha b^2$, so that l is the semi-length of the line which is the "image" of the ellipsoid, we have

$$b^2+\lambda=\frac{h^2-l^2}{\alpha}.$$

The value of Ψ in terms of h thus becomes

$$\Psi=\frac{q\alpha}{\mathrm{K}}\int_h^\infty \frac{dh}{h^2-l^2}. \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Fig. 3.



Equation (11) now becomes

$$\frac{x^2}{h^2}+\frac{\rho^2\alpha}{h^2-l^2}=1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

so that instead of the cylindrical coordinates x and $\rho\left(=\sqrt{y^2+z^2}\right)$

we can take h and ϕ where

$$x=h\cos\phi,\quad \rho=\frac{\sqrt{h^2-l^2}}{\sqrt{\alpha}}\sin\phi. \quad . \quad . \quad . \qquad (19)$$

From (18) we have in terms of h and ϕ

$$\frac{dh}{dx}=\frac{(h^2-l^2)\cos\phi}{h^2-l^2\cos^2\phi},\qquad \frac{dh}{d\rho}=\frac{h\sqrt{h^2-l^2}\sin\phi\sqrt{\alpha}}{h^2-l^2\cos^2\phi}.$$

Hence

$$E_1=-\frac{d\Psi}{dh}\cdot\frac{dh}{dx}=\frac{\alpha q\cos\phi}{K(h^2-l^2\cos^2\phi)}, \quad . \quad . \quad . \quad . \qquad (20)$$

$$E=-\frac{1}{\alpha}\frac{d\Psi}{dh}\cdot\frac{dh}{d\rho}=\frac{qh\sin\phi\sqrt{\alpha}}{K\sqrt{h^2-l^2}\,(h^2-l^2\cos^2\phi)}, \quad . \qquad (21)$$

$$H=KuE_\rho=\frac{quh\sin\phi\sqrt{\alpha}}{\sqrt{h^2-l^2}(h^2-l^2\cos^2\phi)}. \quad . \quad . \quad . \quad . \qquad (22)$$

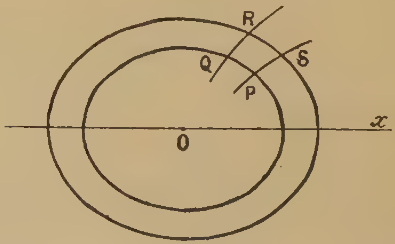
I now pass on to calculate the total energy possessed by the ellipsoid when in motion along its axis of figure. In making the calculation I shall suppose that $a^2>\alpha b^2$, *i. e.*, that l^2 is positive. The case in which $a^2<\alpha b^2$ can be deduced by the appropriate mathematical transformation.

I have shown {§ 22} that the total energy, viz. the volume integral of $\frac{KE^2+\mu H^2}{8\pi}$, due to the motion of a charge on any surface, is

$$W=\tfrac{1}{2}q\Psi_0+2T,$$

where Ψ_0 is the value of the convection-potential at the surface of the body, and T is the magnetic part of the energy, viz., the volume integral of $\mu H^2/8\pi$.

Fig. 4.



Now in fig. 4 let the ellipsoid PQ be determined by h, and the ellipsoid RS by $h+dh$. Let the angular coordinate of P and S be ϕ, and let that of Q and R be $\phi+d\phi$. Then the area PQRS

$$=\frac{d(x,\rho)}{d(h,\phi)}dh\,d\phi=\left(\frac{dx}{dh}\frac{d\rho}{d\phi}-\frac{dx}{d\phi}\frac{d\rho}{dh}\right)dh\,d\phi$$

$$=\frac{h^2-l^2\cos^2\phi}{\sqrt{a}\,\sqrt{h^2-l^2}}dh\,d\phi.$$

Now if the area PQRS revolve about the axis Ox the volume of the ring traced out is

$$2\pi\rho\,\frac{d(x,\rho)}{d(h,\phi)}dh\,d\phi=\frac{2\pi(h^2-l^2\cos^2\phi)\sin\phi}{a}dh\,d\phi.$$

Thus for the magnetic part of the energy we have

$$\mathrm{T}=\iint\frac{\mu\mathrm{H}^2}{8\pi}2\pi\rho\,\frac{d(x,\rho)}{d(h,\phi)}\,dh\,d\phi$$

$$=\frac{\mu q^2u^2}{4}\iint\frac{h^2\sin^3\phi\,dh\,d\phi}{(h^2-l^2)(h^2-l^2\cos^2\phi)}.$$

Since λ goes from 0 to ∞ h goes from a to ∞. The limits of ϕ are 0 and π.

Now

$$\int_0^\pi\frac{\sin^3\phi\,d\phi}{h^2-l^2\cos^2\phi}=-\frac{1}{l^2}\left[\cos\phi-\frac{h^2-l^2}{2hl}\log\frac{h+l\cos\phi}{h-l\cos\phi}\right]_0^\pi$$

$$=\frac{1}{l^2}\left\{2-\frac{h^2-l^2}{hl}\log\frac{h+l}{h-l}\right\}.$$

Hence

$$\mathrm{T}=\frac{\mu q^2u^2}{4l^2}\int_a^\infty\left(\frac{2h^2}{h^2-l^2}-\frac{h}{l}\log\frac{h+l}{h-l}\right)dh$$

$$=\frac{\mu q^2u^2}{4l^2}\left[h-\frac{h^2+l^2}{2l}\log\frac{h+l}{h-l}\right].$$

When h is large the quantity in [ ]

$$=h-l\left(\frac{h^2}{l^2}+1\right)\left(\frac{l}{h}+\frac{l^3}{3h^3}\cdots\right)$$

vanishing when $h=\infty$.

Thus, making use of $\mu\mathrm{K}v^2=1$ we have for the magnetic energy

$$\mathrm{T}=\frac{q^2u^2}{4l\mathrm{K}v^2}\left\{\frac{a^2+l^2}{2l^2}\log\frac{a+l}{a-l}-\frac{a}{l}\right\}.$$

Now by (17) we have at the surface of the ellipsoid

$$\Psi_0=\frac{q\alpha}{\mathrm{K}}\int_a^\infty\frac{dh}{h^2-l^2}=\frac{q\alpha}{2\mathrm{K}l}\log\frac{a+l}{a-l}.$$

Hence the total electromagnetic energy of the ellipsoid is

$$\mathrm{W}=\tfrac{1}{2}q\Psi_0+2\mathrm{T}=\frac{q^2}{4\mathrm{K}l}\left\{\left(1+\frac{u^2a^2}{v^2l^2}\right)\log\frac{a+l}{a-l}-2\frac{u^2a}{v^2l}\right\}. \quad (23)$$

Here we must remember that $l^2=a^2-\alpha b^2$.

(A) *Energy of Heaviside Ellipsoid.* If we put $a/l=\mathrm{S}$ and make S large we have

$$\mathrm{W}=\frac{q^2\mathrm{S}}{2\mathrm{K}a}\left\{\left(1+\frac{u^2\mathrm{S}^2}{v^2}\right)\left(\frac{1}{\mathrm{S}}+\frac{1}{3\mathrm{S}^3}+\ldots\right)-\frac{u^2\mathrm{S}}{v^2}\right\}$$

$$=\frac{q^2}{2\mathrm{K}a}\left(1+\frac{1}{3}\frac{u^2}{v^2}\right)\text{ when }\mathrm{S}=\infty\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

This corresponds to the Heaviside ellipsoid, for when $\mathrm{S}=\infty$ $a^2=\alpha b^2$. The energy of the same ellipsoid at rest is

$$\frac{q^2\sqrt{\alpha}}{2\mathrm{K}a}\cdot\frac{v}{u}\sin^{-1}\frac{u}{v}.$$

(B) *Energy of a Sphere.* Putting $b=a$ we have $l=au/v$, and thus

$$\mathrm{W}=\frac{q^2}{2\mathrm{K}a}\left(\frac{v}{u}\log\frac{v+u}{v-u}-1\right). \quad . \quad . \quad . \quad . \quad (25)$$

If u is small compared with v we have

$$\mathrm{W}=\frac{q^2}{2\mathrm{K}a}\left(1+\frac{2}{3}\frac{u^2}{v^2}+\ldots\right).$$

It will be found that as far as u^2/v^2 the magnetic energy is

$$\frac{q^2u^2}{3\mathrm{K}av^2}=\frac{\mu q^2u^2}{3a}$$

as has been found by Mr. Heaviside*. It follows from this that as far as terms in u^2/v^2 the electric part of the energy is unaltered by the motion.

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* 'Electrical Papers,' vol. ii. p. 505.

(C) *Energy of a very slender Ellipsoid.* When the ellipsoid is so slender that b^2/a^2 may be neglected in comparison with unity we have

$$W = \frac{q^2}{2Ka}\left\{\left(1+\frac{u^2}{v^2}\right)\log\frac{2a}{b\sqrt{1-\frac{u^2}{v^2}}} - \frac{u^2}{v^2}\right\}. \quad . \quad (26)$$

When u/v is small, this becomes

$$W = \frac{q^2}{2Ka}\left\{\left(1+\frac{u^2}{v^2}\right)\log\frac{2a}{b} + \frac{1}{2}\frac{u^2}{v^2}\right\}.$$

(D) *Energy of a Disk.*

When $a^2 < \alpha b^2$ the ellipsoid is more oblate than Heaviside's, and l^2 becomes negative. In this case let us write

$$r^2 = b^2 - \frac{a^2}{\alpha},$$

so that r is the radius of the disk which is the "image" of the ellipsoid a, b. Then writing $\sqrt{-1} = i$ we have from (23)

$$W = \frac{q^2}{4Kir\sqrt{\alpha}}\left(1 - \frac{u^2a^2}{v^2r^2\alpha}\right)\log\frac{1+i\sqrt{\alpha}\,r/a}{1-i\sqrt{\alpha}\,r/a} + \frac{q^2u^2a}{2Kv^2r^2\alpha}.$$

But

$$\frac{1}{i}\log\frac{1+xi}{1-xi} = 2\left(x - \frac{x^3}{3} + \frac{x^5}{5}\ldots\right) = 2\tan^{-1}x,$$

so that (23) becomes

$$W = \frac{q^2}{2Kr\sqrt{\alpha}}\left\{\left(1 - \frac{u^2a^2}{v^2r^2\alpha}\right)\tan^{-1}\frac{r\sqrt{\alpha}}{a} + \frac{u^2a}{v^2r\sqrt{\alpha}}\right\}. \quad (27)$$

When $a = 0$ we find for the energy of a disk of radius r moving along its axis

$$W = \frac{q^2\pi}{4Kr\sqrt{\alpha}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

In all these cases it will be found that when $u = v$ the energy becomes infinite, so that it would seem to be impossible to make a charged body move at a greater speed than that of light.

DISCUSSION.

Prof. PERRY said the paper would help to solve many problems connected with the effect of the rotation of the earth upon electrical surface changes. An expression might be found for the mechanical and magnetic forces due to the motion of a charge at any point of the earth's surface. At the equator a point moves at different velocity at midday to its midnight velocity; it might now be possible to determine the magnetic and mechanical effects due to electric charges at equatorial points.

Mr. BLAKESLEY asked whether, in calculating the mutual action of two charged particles, proceeding at the velocity of light, it was assumed that the lines of motion were parallel.

Mr. SEARLE said he had always considered parallel lines of motion; he could not say whether the force would be zero in any other case. The results arrived at in the paper could be applied to problems connected with distributions of terrestrial charges.

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PROCEEDINGS

AT THE

MEETINGS OF THE PHYSICAL SOCIETY

OF LONDON.

SESSION 1896-97.

February 14th, 1896.

Captain Abney, President, in the Chair.

The following was elected a Member of the Society:—

Mr. G. Halliday.

Prof. Ramsay and Mr. Eumorfopoulos read a paper "On the Determination of High Temperatures by the Meldometer."

Prof. Ramsay showed a Direct-vision Spectroscope of special form.

February 28th, 1896.

Prof. Perry, Vice-President, in the Chair.

The following were elected Members of the Society:—

Messrs. F. Bedell, A. Griffiths, D. Morgan Lewis,
C. Rodgers, and S. Wood.

Sir D. Salomons showed some experiments with Incandescent Lamps.

Prof. Fleming and Mr. Petavel communicated a paper "On an Analytical Study of the Alternating-Current Arc."

March 13th, 1896.

Prof. CAREY FOSTER, Vice-President, in the Chair.

The following were elected Members of the Society:—

Messrs. F. W. HENKEL and D. KORDA.

Mr. J. H. REEVES read a paper on "An addition to the Wheatstone Bridge for the Determination of Low Resistances."

Mr. J. H. REEVES read a note on "The Exact Value of Matthiessen's Standard."

Herr PULUJ communicated a paper on "Kathode Rays," with illustrations.

Mr. W. ACKROYD and Mr. H. B. KNOWLES also exhibited some Röntgen-ray photographs.

March 27th, 1896.

Prof. CAREY FOSTER, Vice-President, in the Chair.

Prof. FLEMING read a paper on "The Edison Effect."

Mr. W. B. MORTON communicated a paper "On the Electromagnetic Effect of Moving Charges."

April 24th, 1896.

Captain ABNEY, President, in the Chair.

Mr. R. A. LEHFELDT communicated a paper on "Symbolism in Thermodynamics."

Mr. ROLLO APPLEYARD read a paper "On the Adjustment of the Kelvin Bridge."

Mr. J. FRITH read a paper "On the Effect of Wave-form on the Alternate-Current Arc."

May 8th, 1896.

Captain ABNEY, President, in the Chair.

The following were elected Members of the Society:—

Messrs. A. THORNTON and E. TREMLETT CARTER.

Messrs. FRITH and RODGERS gave a paper "On the True Resistance of the Electric Arc."

May 22nd, 1896.

Prof. Ayrton, Vice-President, in the Chair.

Mr. Rollo Appleyard read a paper on "Dielectrics."

Prof. Viriamu Jones made a communication "On the Magnetic Field due to an Elliptical Current at a point in the Plane of the Ellipse and within it."

Mr. Campbell read a paper on "New Instruments for the Direct Measurement of the Frequency of Alternating or Pulsating Electric Currents."

June 12th, 1896.

Captain Abney, President, in the Chair.

Mr. Campbell read a paper "On the Measurement of very large and very small Alternating Currents."

Mr. Griffiths exhibited some Improvements in "Apparatus for the Measurement of Electrical Resistance."

Prof. S. P. Thompson made a communication on "Röntgen Rays."

June 26th, 1896.

Captain Abney, President, in the Chair.

The following was elected a Member of the Society :—

Mr. R. H. Sodeau.

Mr. F. Bedell read a paper on "Admittance and Impedance Loci."

Prof. S. P. Thompson read a paper "On the Properties of a Body having Negative Resistance."

Mr. F. W. Burstall gave a paper "On the Use of Bare Wire for Resistance Coils."

Mr. J. Larmor gave a paper "On the Theory of Moving Electrons and Electric Charges."

October 30th, 1896.

Captain Abney, President, in the Chair.

This was a Special Meeting. The following resolutions were passed :—

I. That the Subscription to the Society be increased to £2 2*s.* per annum.
II. That all Members who have already compounded for their subscriptions be invited to contribute to the Funds of the Society voluntarily, by either subscribing One Guinea annually or compounding for such Subscription.
III. That a Guarantee Fund be instituted.
IV. That in future Members of the Society be styled "Fellows of the Physical Society of London."
V. That a Foreign Secretary be appointed who shall be *ex officio* a Member of the Council.
VI. That a Librarian be appointed who shall be *ex officio* a Member of the Council.
VII. That an Editor of the Abstracts be appointed who shall be *ex officio* a Member of the Council.
VIII. That the Articles of Association of the Society be and hereby are altered in the manner to give effect to the previous resolutions.

At the Ordinary Meeting of the same date a letter was read from Lord Kelvin thanking the Society for the Address which the President, on behalf of the Society, had recently presented to him.

The following were elected Members of the Society :—

Messrs. D. B. Mair, G. H. J. Hurst, and F. G. Donnan.

Prof. Stroud and Mr. J. B. Henderson read a paper "On a satisfactory method of Measuring Electrolytic Conductivity by means of Continuous Currents."

Mr. Rollo Appleyard exhibited a number of different forms of Electrical Trevelyan Rockers.

November 13th, 1896.

Captain Abney, President, in the Chair.

The following was elected a Member of the Society :—

Mr. J. W. Gifford.

A communication "On some Experiments with Röntgen's Radiation" was received from Prof. Threlfall and Mr. Pollock.

Mr. Bryan and Dr. Barton gave a paper "On the Absorption of Electric Waves along Wires by a Terminal Bridge."

November 27th, 1896.

Captain Abney, President, in the Chair.

This was a Special Meeting at which the resolutions passed on October 30th were confirmed.

At the Ordinary Meeting of the same date the following were elected Fellows of the Society:—

Miss Fanny Lowater, Messrs. W. E. Harrison and
V. A. Mundella.

Professor Rücker, Vice-President, took the Chair while the President made a communication upon "Apparatus for giving Diagrams of the Efficiency of a Photographic Shutter."

December 11th, 1896.

Prof. Ayrton, Vice-President, in the Chair.

The following were elected Fellows of the Society:—

Messrs. C. Chree and J. E. Myers.

Dr. Chree gave a paper on "Applications of Physics and Mathematics to Seismology."

Mr. R. T. Rudd communicated a paper on "Musical Tubes."

January 22nd, 1897.

Prof. Ayrton, Vice-President, in the Chair.

The following was elected a Fellow of the Society:—

Mr. J. R. Ashworth.

Mr. W. B. Croft gave an exhibition of some simple Apparatus.

Mr. E. C. Baly read a paper "On the Passage of Electricity through Gases."

Annual General Meeting.

February 12th, 1897.

Captain Abney, President, in the Chair.

The following Report of the Council was read by the Chairman :—

In the Annual Report of the Council for the past year there has to be chronicled the greatest change which has taken place in the Society since its first establishment in 1875. With the expansion given to the scope of the work of the Society in the publication every month of the Abstracts, in the printing in advance of proofs of papers to be read before the Society, and in securing a home at Burlington House, it was found impossible to carry on the operations of the Society on a sound and substantial basis under the old regulations. The difficulties were greatly accentuated by the notable decrease in the interest for money invested, compared with the condition in this respect of the money market at the date of the Society's foundation. The Society owes much to the foresight in these financial matters of past Councils, so that on the one hand the necessary changes which the out-going Council have proposed and which the Society has loyally supported them in carrying out, have neither been hurriedly proposed nor postponed till too late, and on the other hand are believed to be sufficient for the purposes of the present day.

Though the annual subscription of Members has been raised, the Council is fully convinced that it is as low as is consistent with maintaining the efficiency of the Society, and is not higher than that of other scientific associations. The formal title of Membership is now conveyed in the word "Fellow."

There have been held fourteen ordinary business meetings since the last report, as well as the two special meetings necessary for the passing and confirming the changes in the constitution of the Society, all at Burlington House.

From a scientific point of view the past year has been marked by the development of Prof. Röntgen's great discovery of the peculiar radiations which by general consent bear his name, and Fellows of the Physical Society have, as might be expected, been forward in these investigations.

A satisfactory number of new members has joined the Society, and the Council feels great satisfaction that only a few resignations have been sent in owing to the changes in the regulations that have been made.

Among the deaths which have taken place are those of Mr. W. H. Walenn, September 20, 1896, and Captain Marshall Hall, April 14, 1896.

WILLIAM HENRY WALENN was born in London, January 7, 1828, and died at his residence, 9 Carleton Road, N., on September 20, 1896, after a long and painful illness. He was trained as an engineer at the works of Messrs. Cottam, and received part of his education at University College, London, where he studied mathematics under Professor De Morgan. He was one of the earliest abridgers of specifications to the Patent Office, beginning under the then Comptroller, Mr. Woodcroft, for whom he compiled the series of abridgments relating to Electricity and Magnetism, Photography, and other subjects. In 1866 his book, 'Little Experiments for Little Chemists,' was published, and in it was given a new process for depositing brass upon zinc. In 1871 he contributed a paper to the 'Philosophical Magazine,' "On the Solutions for Depositing Copper and Brass by means of Electric Force." About this time he conducted some experiments for the Government of the day in electro-depositing copper upon the bottom of an iron ship. Between 1868 and 1880 he communicated through the 'Philosophical Magazine' several mathematical papers upon "Unitates," and methods of checking calculations by means of these. Mr. Walenn was known among his friends as a man always ready and willing to aid any inquirer after scientific truth, whether in his favourite study of mathematics, or in the many branches of physics and chemistry with which he was familiar. He became a Fellow of the Chemical Society in 1866, and of the Institute of Chemistry soon after its establishment.

The adoption of the Report was proposed and passed unanimously.

The election of Officers and other Members of Council then took place, the new Council being constituted as follows :—

President.—Shelford Bidwell, M.A., LL.B., F.R.S.

Vice-Presidents who have filled the Office of President.—Dr. J. H. Gladstone, F.R.S.; Prof. G. C. Foster, F.R.S.; Prof. W. G. Adams, M.A., F.R.S.; Lord Kelvin, D.C.L., LL.D., F.R.S.; Prof. R. B. Clifton, M.A., F.R.S.; Prof. A. W. Reinold, M.A., F.R.S.; Prof. W. E. Ayrton, F.R.S.; Prof. G. F. Fitzgerald, M.A., F.R.S.; Prof. A. W. Rücker, M.A., F.R.S.; Capt. W. de W. Abney, R.E., C.B., D.C.L., F.R.S.

Vice-Presidents.—Major-General E. R. Festing, R.E., F.R.S.; L. Fletcher, M.A., F.R.S.; Prof. J. Perry, D.Sc., F.R.S.; G. Johnstone Stoney, M.A., F.R.S.

Secretaries.—T. H. Blakesley, M.A., M.Inst.C.E.; H. M. Elder, M.A.

Foreign Secretary.—Prof. S. P. Thompson, D.Sc., F.R.S.

Treasurer.—Dr. E. Atkinson, J.P.

Librarian.—C. Vernon Boys, F.R.S.

Other Members of Council.—Walter Baily, M.A.; L. Clark, C.E., F.R.S.; A. H. Fison, D.Sc.; Prof. J. A. Fleming, M.A., F.R.S.; R. T. Glazebrook, M.A., F.R.S.; Prof. A. Gray, M.A.; G. Griffith, M.A.; Prof. G. M. Minchin, M.A., F.R.S.; Prof. W. Ramsay, Ph.D., F.R.S.; J. Walker, M.A.

Votes of thanks were passed to the Chemical Society; to the Officers; and to the Auditors.

THE TREASURER IN ACCOUNT WITH THE PHYSICAL SOCIETY, FROM JANUARY 1ST, 1896, TO DECEMBER 31ST, 1896.

Dr.	£	*s.*	*d.*	£	*s.*	*d.*
Jan. 1, 1896. Balance in the Bank				273	1	9
Entrance-Fees	27	0	0			
Compositions	70	0	0			
Subscriptions for 1890–94	10	0	0			
,, 1895	12	0	0			
,, 1896	175	0	0			
,, 1897	4	4	0			
				298	4	0
Dividends				73	0	8
Grants :—						
British Association	100	0	0			
Royal Society	50	0	0			
				150	0	0
Sales				57	6	8
Advertisements				11	17	10
Deposit Account				6	1	0
Credited by Bank in error				5	1	0
Balance due to the Treasurer				3	4	10
				£877	17	9

Cr.	£	*s.*	*d.*	£	*s.*	*d.*
Jan. 1, 1896. Balance paid to the Treasurer				3	16	2
Taylor and Francis :—						
Separate Copies	19	6	6			
Miscellaneous	50	1	6			
Proceedings	163	6	9			
Abstracts	219	7	3			
				452	2	0
Editor	50	0	0			
,, Petty Cash	4	4	5			
Abstractors	77	13	6			
				131	17	11
Periodicals				16	3	2
Name Plate				2	2	0
Electric Supply				3	3	0
Cheques and Bank charges 6*s.* (less interest 10*d.*)				0	5	2
Attendance				2	4	0
Binding				3	0	0
Reporting	23	10	5			
	6	6	0			
				29	16	5
Excess Subscriptions returned					7	0
Chemical Society	20	9	6			
	22	1	0			
		6	3			
				42	16	9
Stationery					14	9
Petty Cash :—						
Mr. Elder	6	1	2			
Mr. Blakesley		11	4			
Dr. Atkinson	2	9	8			
				9	2	2
Deposit Account				26	1	0
Balance in the Bank, Dec. 31, 1896				154	6	3
				£877	17	9

EDMUND ATKINSON, *Treasurer.*

Examined and found to agree with the vouchers and accounts; also Bankers' Pass Book.

FREDERICK HOVENDEN.
ALFRED W. PORTER. } *Auditors.*

February 1, 1897.

PROPERTY ACCOUNT OF THE PHYSICAL SOCIETY, DECEMBER 31, 1896.

ASSETS.	£	s.	d.
Subscriptions due, estimated	10	0	0
£400 Furness 4 per cent. Debenture Stock	456	0	0
£1000 Midland 4 per cent. Preference Stock	1194	0	0
£200 Metropolitan Board of Works 3½ per cent. Stock	217	0	0
£270 Lancaster Corporation 3 per cent. Stock	227	6	8
£254 2*s.* 9*d.* New South Wales 3½ per cent. Inscribed Stock	249	7	5
Balance in the Bank	154	6	3
Dividend unpaid	4	6	0
Guarantee Fund, on Deposit	100	4	0
	£2612	10	4

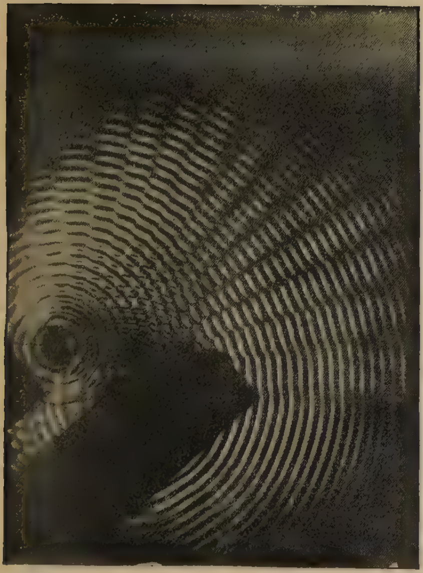
LIABILITIES.	£	s.	d.
Subscriptions for 1897 in advance	4	4	0
Balance due to the Treasurer	3	4	10
Abstracts, Oct.–Dec.	29	5	10
Printers	127	3	9
Balance	2448	11	11
	£2612	10	4

EDMUND ATKINSON, *Treasurer.*

Securities examined and found correct.

FREDERICK HOVENDEN } *Auditors.*
ALFRED W. PORTER. }

February 1, 1897.



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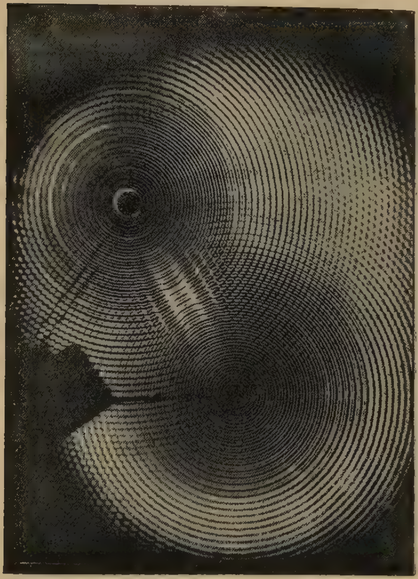

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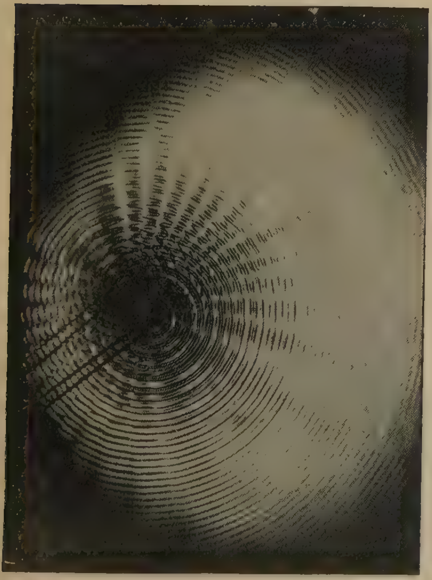

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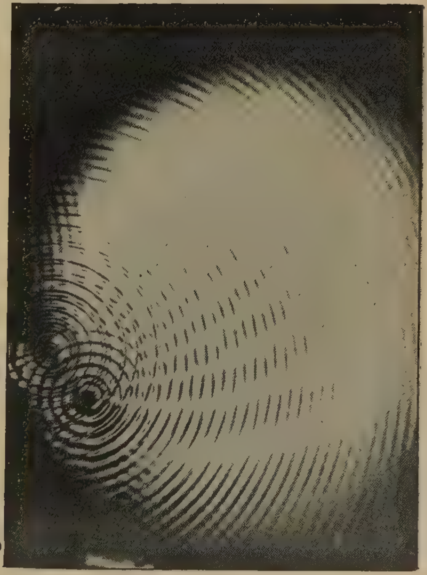

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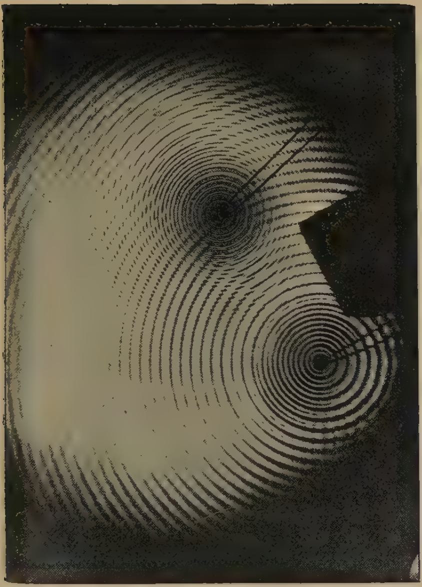

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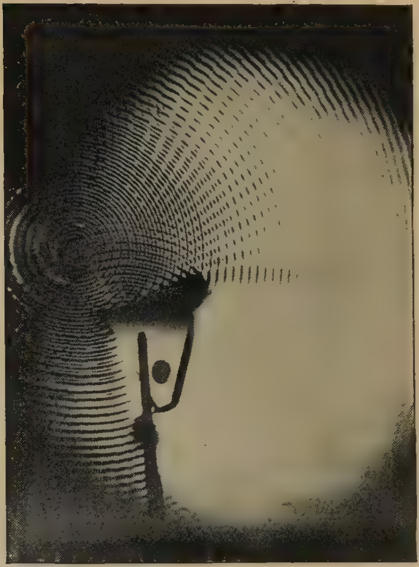

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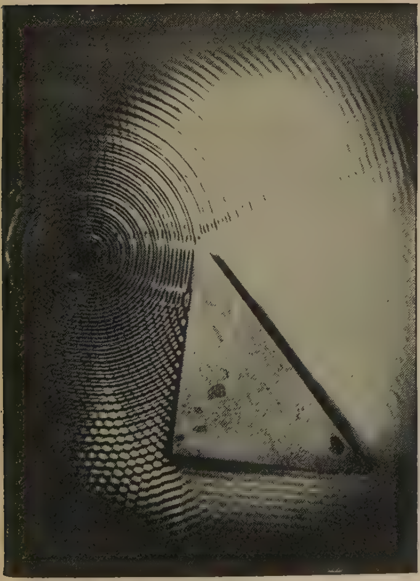

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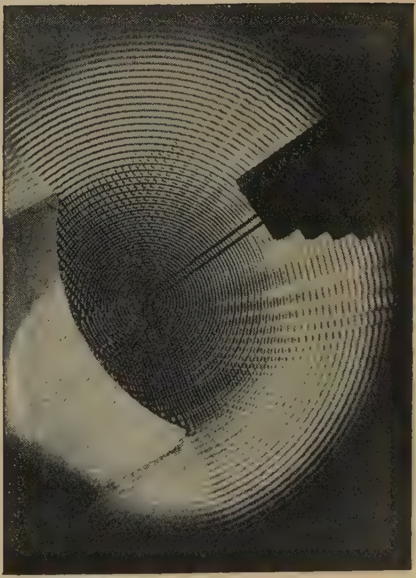

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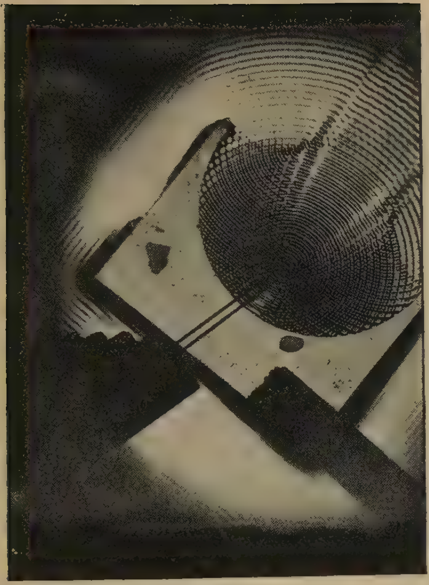

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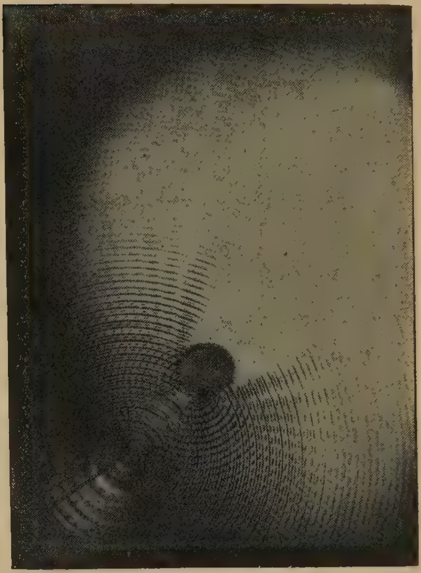
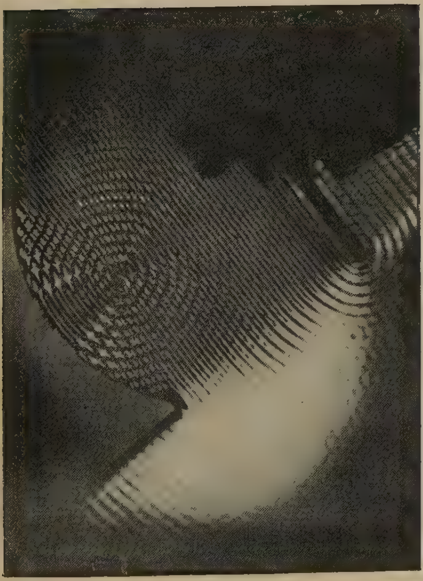


Fig. 10



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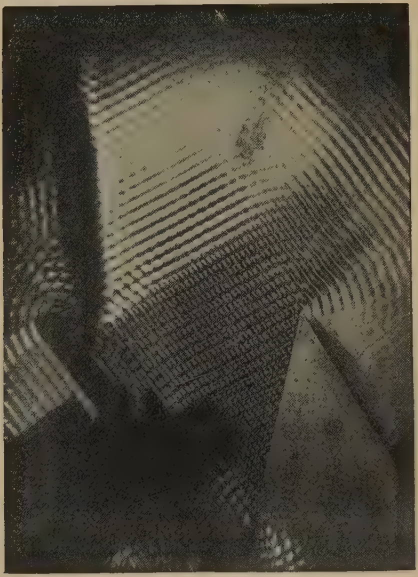
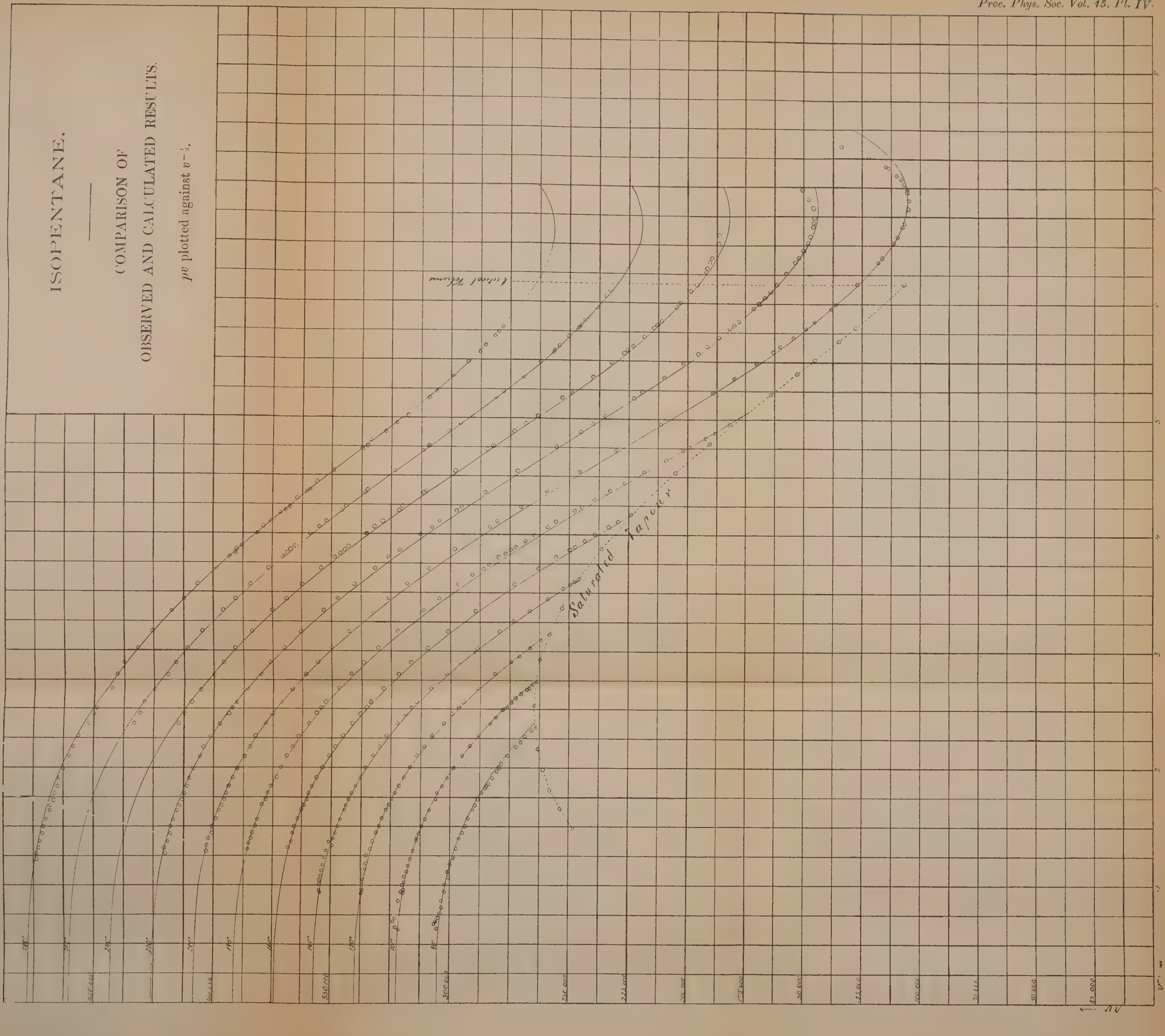


Fig. 12



ISOPENTANE.

COMPARISON OF OBSERVED AND CALCULATED RESULTS.

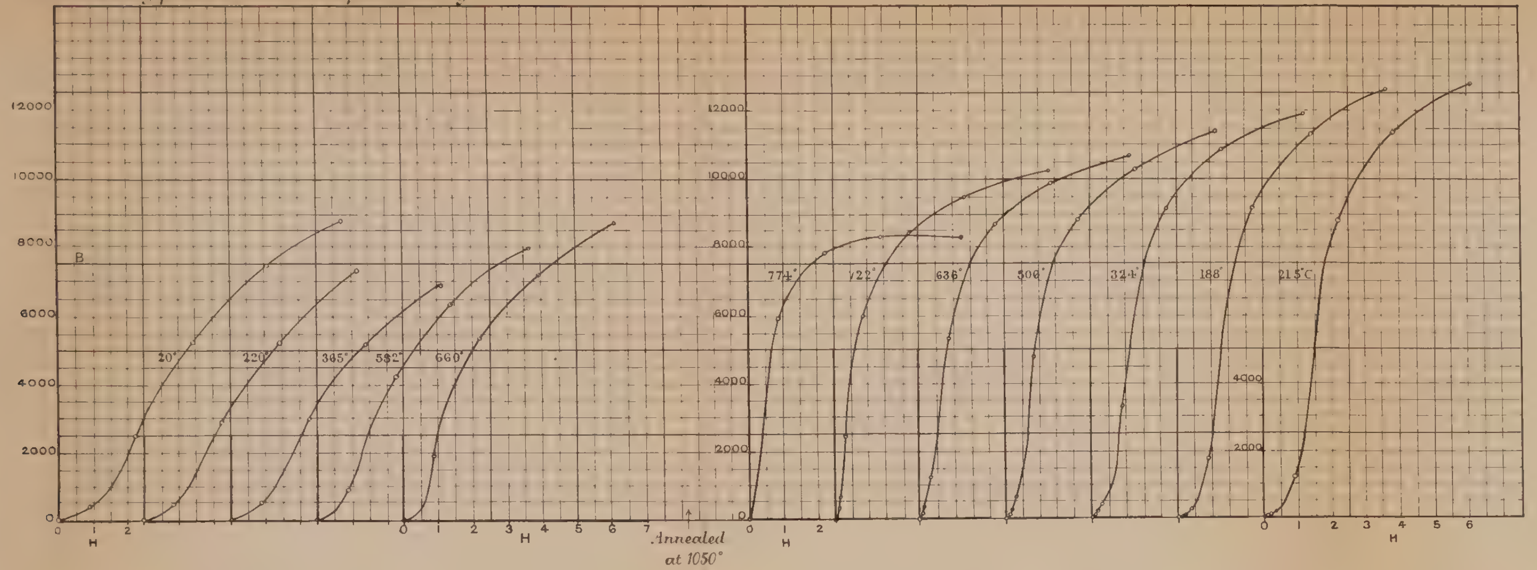
pv plotted against $v^{-\frac{1}{3}}$.

MAGNETIC OBSERVATIONS WITH IRON SPECIMEN A.

Crit. Temp. 795°.

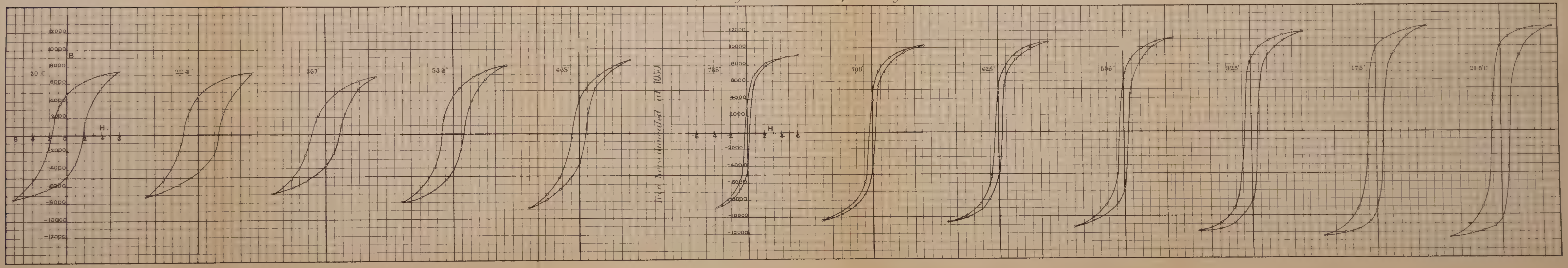
Curves Nº 1. B-H *Curves by the method of reversals*

Annealing process and subsequent cooling



Curves Nº 2. *Hysteresis Loops.*

Annealing Process and subsequent cooling

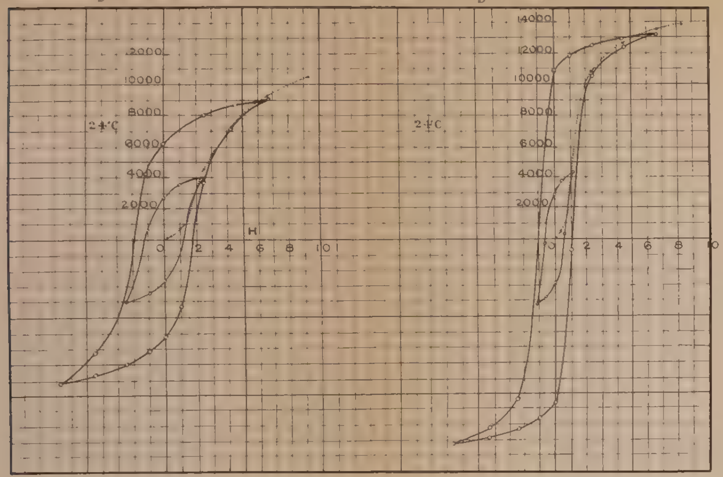


Mintern Bros. lith.

MAGNETIC OBSERVATIONS WITH IRON SPECIMEN B. *(Crit. Temp. 780°)*

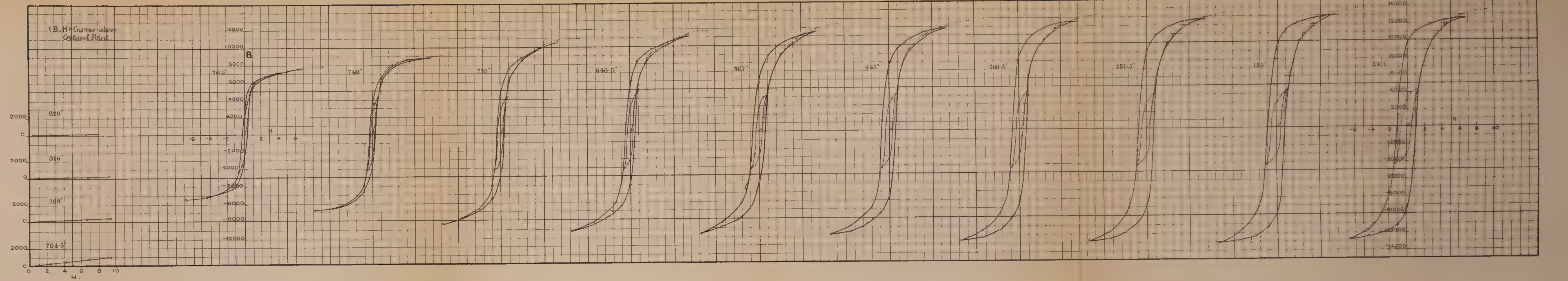
Curves Nº 3.

Preliminary Measurements before and after Annealing at 840°.



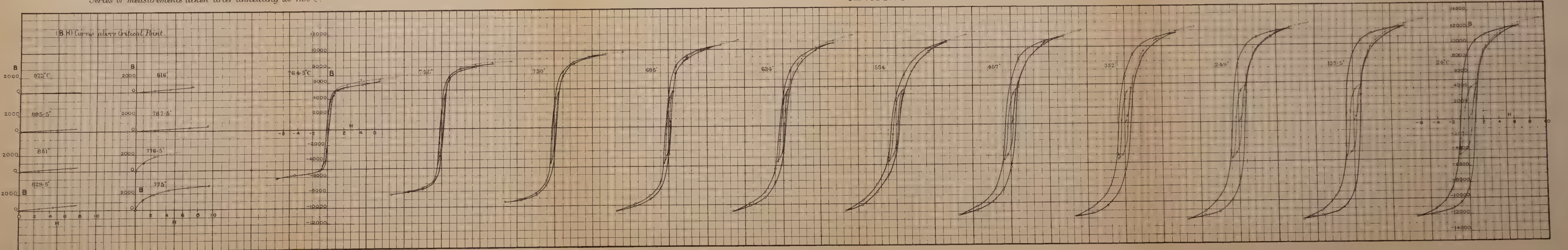
Curves Nº 4.

Series taken after annealing at 840°.



Curves Nº 5.

Series of measurements taken after annealing at 1150°C.



Explanation

The larger Hysteresis Loops are all taken exactly within the Limits H = ± 6·83 *CGS Units.*

The smaller loops are within the limits of Induction B = ± 4260 *lines for the* 1st *series and* B = ± 4550 *lines for the* 2nd *series*

The dotted curves are the B-H *curves taken by the Method of Reversals.*

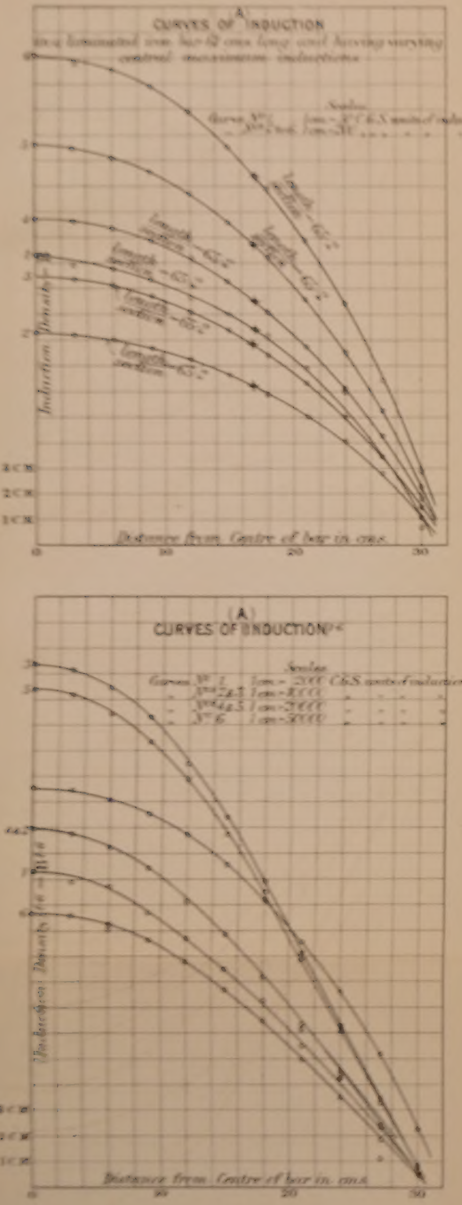


Fig. 1.

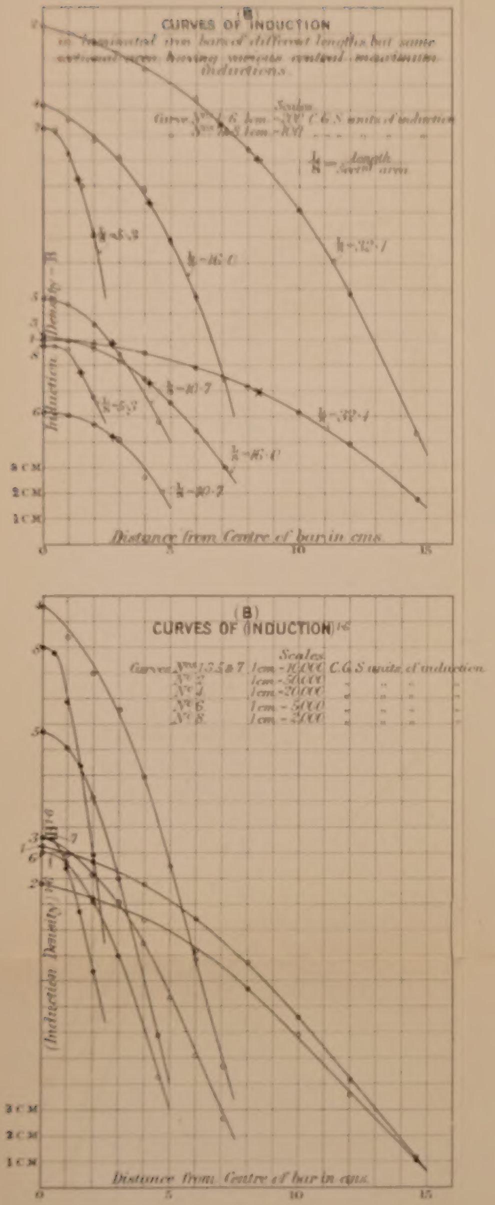


Fig. 2.

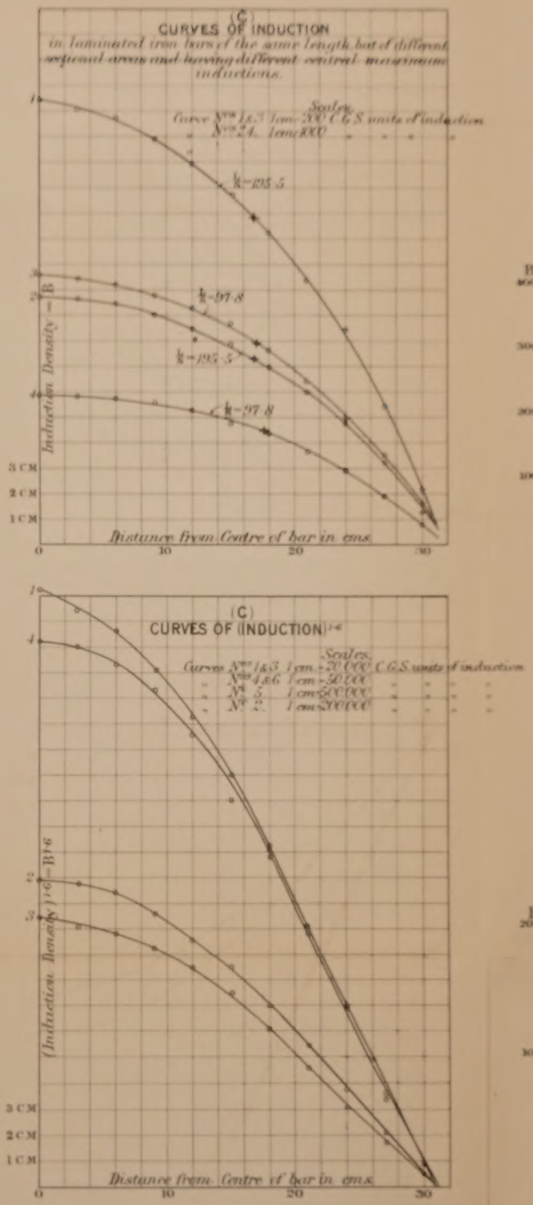
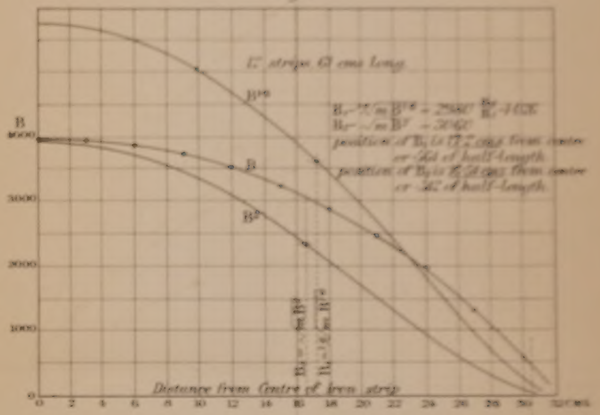


Fig. 3.

Fig 4



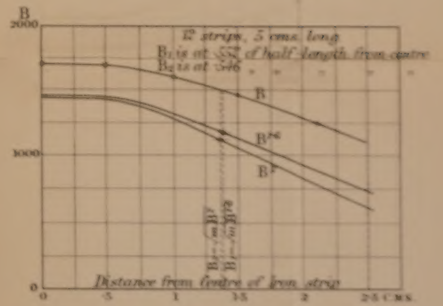


Fig. 5.

Mintern Bros. lith.